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Algebraic Topology I
HS 2014

Problem set 7

Computations in Homology

1. A pair of topological spaces (X, A) is a *good pair* if A is a nonempty closed subspace and there exists an open neighborhood V of A in X that deformation retracts onto A. Prove that, if (X, A) is a good pair, it holds $H_n(X, A) = H_n(X/A)$ for each $n > 0$:

   (a) Let $q : X \to X/A$ be the quotient map. Show that the diagram

   $$\begin{tikzcd} H_n(X,A) \arrow[r] \arrow[d, "q_n"] & H_n(X,V) \arrow[d, "q_n"] & H_n(X-A,V-A) \arrow[l] \arrow[d, "q_n"] \\ H_n(X/A,A/A) \arrow[r] & H_n(X/A,V/A) & H_n(X/A-A/A,V/A-A/A) \arrow[l] \end{tikzcd}$$

   is commutative.

   (b) Show that the horizontal arrows are isomorphisms.

   (c) Show that the vertical arrow on the right is an isomorphism.

   (d) Conclude.

2. Compute the homology of the projective spaces $\mathbb{P}^n(\mathbb{R})$:

   (a) Find a subspace of $\mathbb{P}^n(\mathbb{R})$ homeomorphic to $\mathbb{P}^{n-1}(\mathbb{R})$.

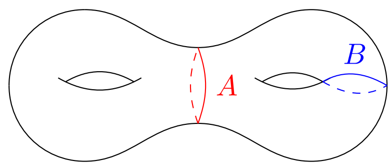
   (b) Show that the pair $(\mathbb{P}^n(\mathbb{R}), \mathbb{P}^{n-1}(\mathbb{R}))$ is a good pair.

   (c) Determine the homotopy type of $\mathbb{P}^n(\mathbb{R})/\mathbb{P}^{n-1}(\mathbb{R})$.

   (d) Compute $H_k(\mathbb{P}^2(\mathbb{R}))$.

   (e) Prove, by induction, that $H_k(\mathbb{P}^n(\mathbb{R}))$ is zero if k is even and bigger than zero, is $\mathbb{Z}/_{2\mathbb{Z}}$ if k is odd and smaller than n, and is $\mathbb{Z}$ if $k = n$ are odd.

3. Compute the relative homology $H_k(\Sigma_2, A)$ and $H_k(\Sigma_2, B)$ where Σ_2 is the surface of genus 2 and A (resp. B) is the curve depicted in red (resp. blue).



4. Let X, Y be CW complexes. Compute the homology of $X \vee Y$ assuming that the homology of X and Y are known.

5. Consider the spaces $X = S^2 \vee S^1 \vee S^1$ and $Y = S^1 \times S^1$. Show that $H_k(X) = H_k(Y)$ for all k, but there exists no homotopy equivalence between X and Y.

6. (a) Show that the quotient map $S^1 \times S^1 \to S^2$ obtained collapsing the subspace $S^1 \vee S^1$ to a point is not null-homotopic. *Hint:* what is the map induced on H_2?

   (b) Show that any map $S^2 \to S^1 \times S^1$ is null-homotopic. *Hint:* use covering spaces.