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Algebraic Topology I
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Solution 2

Fundamental group

1. *Determine which letters of the alphabeth $\{A, B, \ldots, Z\}$ are homeomorphic and which ones are homotopy equivalent.*

   Solution For me a letter of the alphabeth is a 1 dimensional CW complex with a zero cell in each corner. The solution is different if you consider *thick* letters (how?). There are three homotopy equivalence classes

   $$\begin{array}{ll} \cdot & \sim \{C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z\} \\ \circ & \sim \{A, D, O, P, R, \} \\ 8 & \sim \{B, Q\} \end{array}$$

   The fact that they are different can be seen using the fundamental group: the spaces in the first class are contractible hence their fundamental group is trivial, spaces in the second class have fundamental group isomorphic to $\mathbb{Z}$, the letter B (and the letter Q in the font I am using) has fundamental group isomorphic to $\mathbb{F}_2$. Explicit homotopy equivalences can be constructed, or you can use Proposition 0.17 in Hatcher.

   There are many more distinct equivalence classes of homeomorphsm:

segment	$\{C, G, I, J, L, M, N, S, U, V, W, Z\}$
two points of degree 3	$\{H\}$
one point of degree 3	$\{E, F, T, Y\}$
one point of degree 4	$\{K, X\}$
circle	$\{D, O\}$
two points of degree 3	$\{A, R\}$
one point of degree 3	$\{P\}$
	$\{B\}$
	$\{Q\}$

Explicit homeomorphisms between those topological spaces can be constructed, and it is possible to check that the classes are distinct using the fact that there exist special points of difference valency, that hence locally disconnect the space in a different number of connected components.

2. *Prove that if two path connected spaces X, Y are homotopy equivalent, then for every $x \in X$, $y \in Y$ the groups $\pi_1(X, x)$ and $\pi_1(Y, y)$ are isomorphic.*

   Solution One should first prove that if $f : X \to X$ is homotopic to the identity, then $f_* : \pi_1(X, x) \to \pi_1(X, f(x))$ is an isomorphism. In fact if H is an homotopy between f and the identity, and if we denote by $h : [0, 1] \to X$ the path given by $h(t) = H(x, t)$, we get that $f_* = \beta_h : \pi_1(X, x) \to \pi_1(X, f(x))$ since it is possible to use the homotopy H to construct an homotopy between $f \circ \gamma$ and $\beta_h(\gamma)$ for every loop $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1) = x$.

   This implies that if X and Y are homotopy equivalent and $g : X \to Y$, $z : Y \to X$ are maps such that $g \circ z : Y \to Y$ is homotopic to the identity of Y and $z \circ g$ is homotopic to the identity of X, one gets that $g_* \circ z_* = (g \circ z)_* : \pi_1(X, x) \to \pi_1(X, g \circ z(x))$ is an isomorphism, hence in particular z_* is injective and g_* is surjective. Similarly $z_* \circ g_*$ is an isomorphism and hence we get that g_* is injective. This implies that g_* is a group homomorphism that is injective and surjective, hence it is an isomorphism.

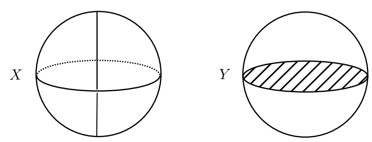
3. *Let X be a path connected space. Show that $\pi_1(X)$ is abelian if and only if all basepoint change homomorphisms β_h depend only on the endpoints of the path h.*

   Solution Let h be a closed path based at x_0, we have by definition that $\beta_h([\gamma]) = [h \cdot \gamma \cdot \overline{h}] = [h][\gamma][\overline{h}]$. Moreover, if we assume that the basepoint change homomorphism depends only on the endpoints of the path h, we get that the basepoint change homomorphism with respect to a the closed loop h is the identity (since this is true for an homotopically trivial loop). Let us consider two elements $[\gamma_1], [\gamma_2]$ in $\pi_1(X, x_0)$. We have $[\gamma_2\gamma_1\overline{\gamma}_2] = \beta_{g_2}([\gamma_1]) = [\gamma_1]$ and hence in particular $[\gamma_2][\gamma_1] = [\gamma_1][\gamma_2]$.

Viceversa assume that $\pi_1(X, x_0)$ is abelian and fix two different paths h_1, h_2 with the same endpoints x_0, x_1, we want to show that for every element $[\gamma] \in \pi_1(X, x_0)$ then $\beta_{h_1}([\gamma]) = \beta_{h_2}([\gamma]) \in \pi_1(X, x_1)$. Let us consider the closed loop $c = h_2\overline{h}_1$, we have

$$\begin{aligned}
[h_1\gamma\overline{h}_1] &= [c][\overline{c}]h_1 g\overline{h}_1] \\
&= [c][h_1 g\overline{h}_1][\overline{c}] \\
&= [h_2\overline{h}_1 h_1 g\overline{h}_1 h_1\overline{h}_2] \\
&= [h_2 g\overline{h}_2].
\end{aligned}$$

4. *Compute the fundamental groups of the spaces X, Y of exercise 3, problem set 1.*



Solution We know that X is homotopy equivalent to $S^1 \vee S^2$ and Y is homotopy equivalent to $S^2 \vee S^2$. The same argument as in the proof of [Hatcher, Proposition 1.14] implies that Y is simply connected, and $\pi_1(X) = \pi_1(S^1) = \mathbb{Z}$.

5*. *Let A be a path connected space, and X is obtained from A attaching cells e^n with $n \geq 2$. Prove that the inclusion $A \hookrightarrow X$ induces a surjection on fundamental groups.*

Solution Let us fix a basepoint x in A, and let us consider a class $\gamma \in \pi_1(X, x)$. It is enough to show that any representative g of γ is homotopic relative to x to a path g' whose image is contained in A. In order to do this we can assume, by induction, that X is obtained from A attaching a single n cell e_n, and, by the same argument as in the proof of [Hatcher, Proposition 1.14], that there exists a point y in e_n that doesn't belong to the image of g. We can then use the deformation retraction of $e_n \setminus \{y\}$ onto its boundary to get an homotopy of g to a new path g' that is homotopic to g and whose image is contained in A.

Find a set of generators for the fundamental group of a CW complex with a single 0 cell.

Solution A set of generators is given by the one cells that form the one skeleton of the CW complex.

6. *Does the Borsuk-Ulam theorem hold for the torus? In other words is it true that for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ there exists a pair $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?*

   Solution No. Consider for example the map $f : S^1 \times S^1 \to \mathbb{R}^2$ defined by $f(x, y) = j(x)$ where we denote by $j : S^1 \to \mathbb{R}^2$ the canonical inclusion as the circle of radius 1. Then for every pair $(x, y) \in S^1 \times S^1$, we have $f(x, y) \neq f(-x, -y)$.

7*. *Let A_1, A_2, A_3 be compact sets in $\mathbb{R}^3$. Use the Borsuk-Ulam theorem to show that there is one plan $P \subseteq \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.*

   Solution Since A_1 is compact, for each unit vector v in S_2 there exists an affine plane P_v in $\mathbb{R}^3$ that is orthogonal to v and with the property that P_v divides A_1 in two pieces of equal measure. Moreover we can assume that the association $v \to P_v$ is continuous by requiring that P_v is the hyperplane with this property that has smallest distance from zero. For each plane P_v we consider the halfspace H_v of $\mathbb{R}^3$ that is bounded by P_v is in the direction pointed by v.

   Let us now consider the function $f : S^2 \to \mathbb{R}^2$ with

   $$f(v) = (m(H_v \cap A_2), m(H_v \cap A_3)).$$

   The function f is continuous. As a consequence of Borsuk Ulam's Theorem we get that there exists a point $v \in S^2$ with $f(v) = f(-v)$ this means that for each A_i the hyperplane P_v divides the compact set A_i in two sets of equal measure, and this concludes the proof.