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Algebraic Topology I
HS 2014

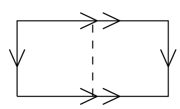
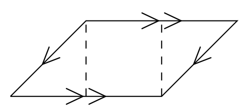
Solution 4

Covering spaces- Homology

1. Use covering theory to prove the Nielsen Schreier Theorem: every subgroup of the free group $\mathbb{F}_2$ is free.

   Solution. $\mathbb{F}_2$ is the fundamental group of the topological space $S^1 \vee S^1$. Let G be a subgroup of $\mathbb{F}_2$, by the fundamental theorem of covering spaces there exists a covering X of $S^1 \vee S^1$ such that $G = \pi_1(X, x)$. Since X is a 1 dimensional CW complex we get that its fundamental group is free, and this concludes the proof.

2. Let us consider the covering spaces of the torus $X = T^2$:

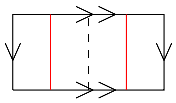
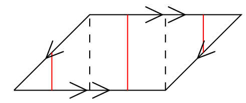
   $\overline{X}_1$  $\overline{X}_2$ 

   (a) Show that $\overline{X}_i$ are homeorphic topological spaces, and the described projections give covering spaces of the torus. Determine the number of sheets.

   Solution: Both topological spaces $\overline{X}_1$ and $\overline{X}_2$ are homeomorphic to the torus, in particular they are homeomorphic. It is easy to check that the projections give two sheeted covering spaces of the torus.

   (b) Find a loop in X whose lifts in $\overline{X}_1$ and $\overline{X}_2$ are not homeomorphic. Deduce that $\overline{X}_1$ and $\overline{X}_2$ are not isomorphic as covering spaces.

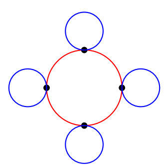
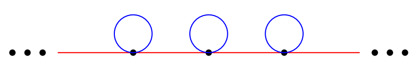
   Solution: We consider a vertical loop in the torus. The lift in $\overline{X}_1$ consists of the disjoint union of two circles, the lift in $\overline{X}_2$ is connected. In particular the lifts cannot be homeomorphic. If instead the two coverings were isomorphic as covering spaces, we would get that the lifts of any path would be homeomorphic.

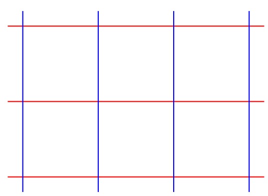
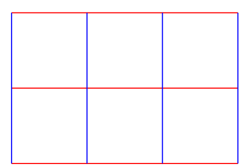
$\overline{X}_1$  $\overline{X}_2$ 

(c*) To which subgroups of $\mathbb{Z}^2 = \pi_1(X, x)$ correspond the two covering spaces?

Solution the first space correspond to the subgroup generated by the elements $(2, 0)$ and $(0, 1)$, the second to the subgroup generated by the elements $(2, 0)$ and $(1, 1)$.

3. Find the subgroups of $\mathbb{F}_2$ corresponding to the following coverings of $S^1 \vee S^1$:

X_1  X_2 

X_3  X_4 

The space X_3 is understood to be extending infinitely in all four directions (it is a grid in $\mathbb{R}^2$, in the graph X_4 the two vertical outermost segments and the two horizontal outermost segments are identify: it is a graph drawn on the torus with two horizontal segments and three vertical segments.

Solution If a denotes the generator of the fundamental group of $S^1 \vee S^1$ associated to the red loop, and b denotes the one associated to the blue loop we have:

(a) X_1 is associated to $\langle a^4, b, aba^{-1}, a^2ba^{-2}, a^3ba^{-3}\rangle$ and consists of all words words $a^{\alpha_1}b^{\beta_1} \dots a^{\alpha_k}b^{\beta_k}$ with $\sum \alpha_i = 0$ modulo 4;

(b) X_2 is associated to $\langle a^k ba^{-k} | k \in \mathbb{Z}\rangle$ and consists of all words words $a^{\alpha_1}b^{\beta_1} \dots a^{\alpha_k}b^{\beta_k}$ with $\sum \alpha_i = 0$;

(c) X_3 is associated to the subgroup consisting of words $a^{\alpha_1}b^{\beta_1} \dots a^{\alpha_k}b^{\beta_k}$ with $\sum \alpha_i = \sum \beta_i = 0$

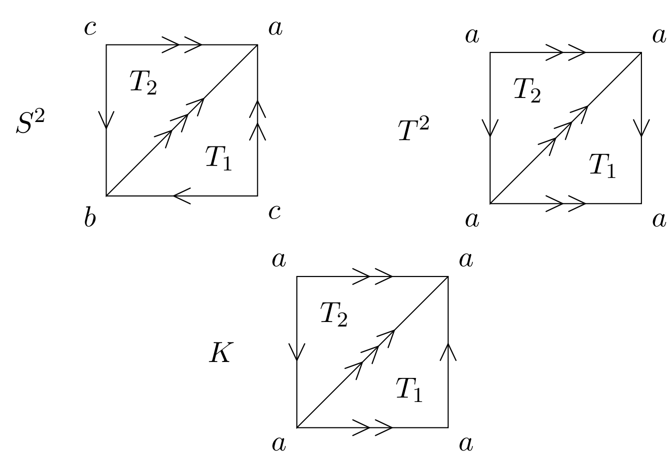
(d) X_3 is associated to the subgroup consisting of words $a^{\alpha_1}b^{\beta_1}\dots a^{\alpha_k}b^{\beta_k}$ with $\sum \alpha_i = 0$ modulo 3 and $\sum \beta_i = 0$ modulo 2.

In order to check that the answer is correct, it is enough to check which loops in $S^1 \vee S^1$ lift to closed loops in X_i, those will correspond to the subgroup of $\mathbb{F}_2$ associated to the covering. In order to do that it is enough to number the elements in the fiber of the basepoint and understand what is the endpoint of the lift of a and b based at any given preimage of the fiber. This is easier since all coverings are normal.

4. Denote by S^2 the two sphere, by T^2 the torus $S^1 \times S^1$, by K the Klein bottle.

   (a) Define a Delta complex structure on S^2, T^2, K with two top dimensional simplices.

   Solution All spaces can be obtained from the square by identifying its sides. The sphere admits a Δ-complex structure with 3 0-simplices $\{a, b, c\}$, 3 1-simplices, 2 2-simplices. The Torus and the Klein bottle have a single 0 simplex, $\{a\}$, three 1-simplices and 2 2-simplices. In all the pictures the 1-simplices are oriented as indicated by the arrows, and the two simplices are oriented so that the first vertex v_0 is sent to the unique vertex with two outcoming arrows, and the last vertex is sent to the one with two incoming arrows.



(b) How many top dimensional simplices do you need at least in a simplicial structure on S^2?

Solution: You need at least 4: indeed if we fix one vertex of a simplicial structure on the sphere there must be at least three 2 dimensional simplices having the vertex as a vertex (because in a simplicial complex two simplices cannot share more than one face), since the simplicial complex with 3 2-dimensional simplices with one vertex in common is not homeomorphic to the sphere but to the disc, you need at least 4 simplices to construct a simplicial complex structure on the sphere. The boundary of the 3 simplex is a simplicial complex with 4 two dimensional simplices that is homeomorphic to a sphere.

(c) Compute the Delta complex homology for the three spaces.

Solution: It is easy to determine the face operators with respect to the given structures.

i. For the sphere we get $\mathrm{C}_0^\Delta(S^2) = \mathbb{Z}^3$ with generators a, b, c, $\mathrm{C}_1^\Delta(S^2) = \mathbb{Z}^3$ with generators e_1, e_2, e_3 (where the index denotes the number of arrows on the 1 simplices in the picture, $\mathrm{C}_2^\Delta(S^2) = \mathbb{Z}^2$ with generators T_1, T_2. The boundary homomorphisms are

$$\partial_2(x_1T_1 + x_2T_2) = (x_1 + x_2)e_1 + (-x_1 - x_2)e_2 + (x_1 + x_2)e_3)$$
$$\partial_1(x_1e_1 + x_2e_2 + x_3e_3) = (x_2 + x_3)a + (x_1 - x_3)b + (-x_1 - x_2)c$$

This implies that $H_2^\Delta(S^2) = \ker \partial_2 = \mathbb{Z}$ generated by $T_2 - T_1$, $H_1^\Delta(S^2) = 0$ since $\ker \partial_1 = x_1e_1 - x_1e_2 + x_1e_3 = \partial_2(x_1T_2)$, and $H_0^\Delta(S^2) = \mathbb{Z}$ since $\mathrm{im}\partial_1 = x_a a + x_b b - (x_a + x_b)c$ and hence $\mathbb{Z}^3/_{\mathrm{im}\partial_1} = \mathbb{Z}$.

ii. For the torus we get $\mathrm{C}_0^\Delta(T^2) = \mathbb{Z}$ with generator a, $\mathrm{C}_1^\Delta(T^2) = \mathbb{Z}^3$ with generators e_1, e_2, e_3 (where the index denotes the number of arrows on the 1 simplices in the picture, $\mathrm{C}_2^\Delta(T^2) = Z^2$ with generators T_1, T_2. The boundary homomorphisms are

$$\partial_2(x_1T_1 + x_2T_2) = (x_1 + x_2)e_1 + (-x_1 - x_2)e_2 + (x_1 + x_2)e_3$$
$$\partial_1(x_1e_1 + x_2e_2 + x_3e_3) = 0$$

This implies that $\mathrm{H}_2^\Delta(T^2) = \ker \partial_2 = \mathbb{Z}$ generated by $T_2 - T_1$, $\mathrm{H}_1^\Delta(T^2) = \mathbb{Z}^2$ since $\mathrm{im}\partial_2 = ae_1 - ae_2 + ae_3$, and $\mathrm{H}_0^\Delta(T^2) = \mathbb{Z} = \mathrm{C}_0^\Delta(T^2)$.

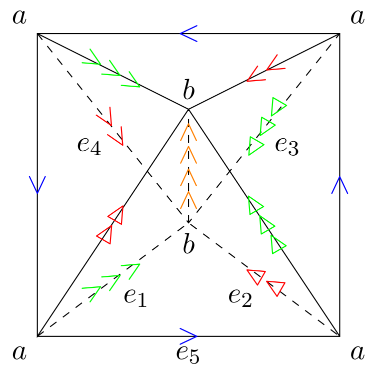
iii. For the Klein Bottle we get $\mathrm{C}_0^\Delta(K) = \mathbb{Z}$ with generator a, $\mathrm{C}_1^\Delta(K) = \mathbb{Z}^3$ with generators e_1, e_2, e_3 (where the index denotes the number of arrows on the 1 simplices in the picture, $\mathrm{C}_2^\Delta(K) = \mathbb{Z}^2$ with generators T_1, T_2. The boundary homomorphisms are

$$\partial_2(x_1T_1 + x_2T_2) = (x_1 + x_2)e_1 + (x_1 - x_2)e_2 + (-x_1 + x_2)e_3$$
$$\partial_1 = 0$$

This implies that $H_2^\Delta(K) = 0$ since ∂_2 is injective, and $H_0^\Delta(S^2) = \mathbb{Z}$ since $\partial_1 = 0$. It is a bit more complicated to check that $H_1^\Delta(K) = \mathbb{Z} \times \mathbb{Z}/_{2\mathbb{Z}}$. In fact we have $\ker \partial_1 = \mathbb{Z}^3$ and we need to compute the quotient for the subgroup $\mathrm{im}(\partial_2) = \mathbb{Z}(1, 1, -1) + \mathbb{Z}(1, -1, 1)$. We consider the surjective homomorphism $h : \mathbb{Z}^3 \to \mathbb{Z} \times \mathbb{Z}/_{2\mathbb{Z}}$ given by $(a, b, c) \mapsto (b + c, a - 1)$ it is easy to check that $\mathrm{im}(\partial_2)$ is contained in the kernel of h, and with a bit more work one can verify that the two subgroups coincide.

5*. Construct a three dimensional Δ-complex X from 4 tethrahedra $T_1, \ldots, T_4$ by first arranging the tethrahedra in a cyclic pattern as in the figure so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show that the Δ-complex homology groups of X are $H_0^\Delta(X) = \mathbb{Z}, H_1^\Delta(X) = \mathbb{Z}/m\mathbb{Z}, H_2^\Delta(X) = 0, H_3^\Delta(X) = \mathbb{Z}$.

Solution



In the Δ complex structure we have

(a) 2 0-simplices a, b;

(b) 6 1-simplices of which four $e_1, \ldots, e_4$ are denoted with red and green arrows, the last e_5 corresponds to the blue arrow and e_6 to the orange one;

(c) 8 2 simplices (four internal faces f_i, where the face f_i has boundary $e_i + e_6 - e_{i+1}$ and four external ones s_i, where the face s_i has boundary $e_5 + e_{i+1} - e_i$);

(d) 4 2 simplices T_i with boundary $-f_i + f_{i+1} - s_i + s_{i+1}$

It is easy to check that the kernel of ∂_3 is generated by the sum $\sum T_i$, hence in particular $H_3^\Delta(X) = \mathbb{Z}$, and that, since the image of ∂_1 are the elements of the form $na - nb$ we get $H_0^\Delta = \mathbb{Z}$.

In order to verify that $H_2^\Delta(X)$ is zero one needs to check that the kernel of ∂_2 equals the image of ∂_3.

Let now $\alpha = \sum a_i f_i + b_i s_i$ be an element of the kernel of ∂_2. We have

$$\partial_2(\alpha) = \sum (a_i - a_{i-1} - b_i + b_{i-1}) e_i + (a_1 + a_2 + a_3 + a_4) e_5 + (b_1 + b_2 + b_3 + b_4) e_6.$$

From the fact that the coefficients of e_5 and e_6 must be zero we get that α must be a linear combination of $s_1 - s_2, s_3 - s_2, s_4 - s_3, f_1 - f_2, f_3 - f_2, f_4 - f_3$, from the equation on the coefficient of e_i we get that the coefficient of $s_i - s_{i+1}$ is equal to the one of $f_i - f_{i+1}$. In particular denoting by x_i this coefficient this implies that $\alpha = \partial_3 \sum x_i T_i$.

We now want to verify that $H_1^\Delta(X)$ is $\mathbb{Z}/_{4\mathbb{Z}}$ generated by the loop e_5. It is immediate to check that $\ker(\partial_1)$ is isomorphic to $\mathbb{Z}^5$ generated by $e_5, e_6, e_2 - e_1, e_3 - e_2, e_4 - e_3$. Let us consider the homomorphism $\phi : \mathbb{Z}^5 \to \mathbb{Z}/_{4\mathbb{Z}}$ given by $(a_1, \ldots, a_5) \mapsto \sum a_i$. The homomorphism is clearly surjective, moreover it follows from the expressions given for ∂_2 that $\mathrm{im}\partial_2 < \ker\phi$. In order to show that the two group are equal it is enough to notice that $e_5 = e_i - e_{i+1} - \partial_2(s_i)$, $e_5 = \partial_2(s_i + f_i) - e_6$, $e_5^4 = \partial(f_1 + f_2 + f_3 + f_4)$.

$$
\begin{aligned}
&\mathrm{H}_3^\Delta(X) = \mathbb{Z} = (1,1,1,1)\mathbb{Z}\\
&\mathrm{H}_2^\Delta(X) = 0\\
&\mathrm{H}_1^\Delta(X) = \mathbb{Z}/_{4\mathbb{Z}}\\
&\mathrm{H}_0^\Delta(X) = \mathbb{Z} = \mathbb{Z}^2/_{(x,-x)\mathbb{Z}}
\end{aligned}
$$