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Algebraic Topology I
HS 2014

Solution 6

Homology, Hurewitz homomorphism

1. Prove that if R is a retract of X then each group $H_n(X)$ contains a subgroup isomorphic to $H_n(R)$ and, in fact, $H_n(X)$ can be expressed as the product $H_n(X) \cong H_n(R) \times K_n$. Give an example of a retract in which K_n is nonzero for some n.

   Solution: Let us consider the long exact sequence of the pair (X, R):

$$\longrightarrow H_n(R) \underset{t_n}{\overset{s_n}{\rightleftarrows}} H_n(X) \longrightarrow H_n(X,R) \longrightarrow H_{n-1}(R)$$

   The map t_n is induced by the inclusion t of R in X, and the map s_n by the retraction $s : X \to R$. Since $s \circ t = \mathrm{Id}_X$ we get that the same holds for the map induced in homology. Since this implies that t_n is injective, we get that the connection map $\partial : H_n(X,R) \to H_{n-1}(R)$ is the zero map, and we have short exact sequences

$$0 \longrightarrow H_n(R) \underset{t_n}{\overset{s_n}{\rightleftarrows}} H_n(X) \longrightarrow H_n(X,R) \longrightarrow 0$$

   Therefore t_n gives an isomorphism of $H_n(R)$ with the subgroup $t_n(H_n(R))$ of $H_n(X)$. In order to finish the proof it is enough to notice that $K_n = \ker(s_n)$ is a subgroup of $H_n(X)$ isomorphic to $H_n(X,R)$, and $H_n(X)$ splits as the product $K_n \times H_n$ since both groups are normal (since $H_n(X)$ is commutative) and their intersection is trivial (by exactness).

   Examples are given by the pairs $(S^n, \{pt\})$ or $(T^2, S^1 \times \{0\})$: in both pairs the smallest subspace is a retract, and in both cases it follows from the computations done in class that the space $K_n \cong H_n(S^n, \{\mathrm{pt}\})$ in the first case and the spaces $K_2 \cong H_2(T_2, S^{\times}\{0\}), K_1 \cong H_1(T_2, S^1 \times \{0\})$ in the second are not trivial.

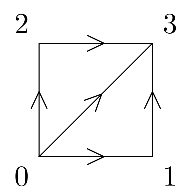
2. A singular 1-simplex $\sigma : \Delta^1 \to X$ is called a *loop* if $\sigma(v_0) = \sigma(v_1)$.

   (a) Prove that a loop is a 1-cycle.

   Solution: This is easy: $\partial_1(\sigma) = \sigma(v_1) - \sigma(v_0) = 0$.

   (b) Two loops σ_0, σ_1 are *freely homotopic* if there exists a continuous map $F : [0,1]^2 \to X$ such that $F(t,0) = \sigma_0(t), F(t,1) = \sigma_1(t), F(0,s) = F(1,s)$. Prove that two freely homotopic loops are homologous, and that the constant path σ_2 is null homologous.

   Solution: Let us consider the picture



   and the singular chain $f_1 - f_2$ where $f_i : \Delta^2 \to X$ is obtained by composing F with the identification of Δ^2 with the simplex in the square with vertices $\{0, i, 3\}$. It is easy to check that, since $F(0,s) = F(1,s)$, we get $\partial_2(f_1 - f_2) = \sigma_0 - \sigma_1$, and hence the two loops are homotopic.

   In order to check that the constant map at the point x is null homologous let us consider the constant 2 simplex $f : \Delta^2 \to X$, $f(t) = x$. It is easy to check that $\partial_2 f = \sigma_2$, hence the statement.

   (c) Fix a basepoint $x \in X$. Define a map $\rho : \pi_1(X,x) \to H_1(X)$ and prove that it is a well defined homomorphism. Deduce that there exists a map $\overline{\rho} : \pi_1(X,x)^{\mathrm{ab}} \to H_1(X)$.

   Solution: We define the map ρ by associating to a loop in the fundamental group, the corresponding one cycle. The map ρ is well defined since we proved that two freely homotopic loops are homologous, and in particular two loops homotopic relative to x are homologous. In order to check that the map is an homomorphism, it is enough to check that, if σ_0, σ_1 are loops based at x, then the loop associated to the concatenation $\sigma_0 \cdot \sigma_1$ is homologous to the sum of the loops σ_0 and σ_1. This last fact is true because $\sigma_0 + \sigma_1 - \sigma_0 \cdot \sigma_1$ is the boundary of the simplex $f : \Delta^2 \to X$

defined by $f(x,y) = \sigma_0 \cdot \sigma_1(x)$ where we parametrize the standard 2-simplex as $\Delta^2 = \{(x,y) | x \in [0,1], y \in [0, \min\{x, 1-x\}]\}$ in such a way that $v_0 = (0,0), v_1 = (0.5, 0.5), v_2 = (1,0)$.

Since $H_n(X)$ is an abelian group, the homomorphism ρ factors through the abelianization of $\pi_1(X,x)$.

(d) A 1-chain $\sigma_0 + \ldots + \sigma_{r-1}$ with $\sigma_i(v_0) = \sigma_{i-1}(v_1)$ for all $i \in \mathbb{Z}/_{r\mathbb{Z}}$ is called an *elementary 1-cycle*. Prove that an elementary 1-cycle is a 1-cycle, and it is homologous to a loop.

Solution: The same proof as in (b) gives that, if σ_a and σ_b are chains corresponding to arcs with the additional property that $\sigma_a(1) = \sigma_b(0)$, then the chain $\sigma_a + \sigma_b$ is homologous to the chain given by the concatenation $\sigma_a \cdot \sigma_b$. It is worth remarking that this is not a direct consequence of part (b) since the chains $\sigma_a, \sigma_b, \sigma_a \cdot \sigma_b$ do not, in general, define classes in $H_1(X)$, since they are not closed in general.

(e) Prove that the classes of loops generate $H_1(X)$.

Solution: It is easy to prove, by induction of the number of simplices that a cycle in $C_1(X)$ is a sum of elementary 1-cycle (details are carried out in Hatcher, page 167).

(f) Assume that X is path-connected. Show that ρ is surjective.

Solution: In order to check this, it is enough to check that each loop σ belongs to the image of ρ. Let us fix a loop σ and a path γ with $\gamma(0) = x$ and $\gamma(1) = \sigma(0)$. The concatenation $\gamma \cdot \sigma \cdot \overline{\gamma}$ is homologous to σ and belongs to the image of ρ.

3. Let $f : (X, x_0) \to (Y, y_0)$ be a map of pointed spaces and consider the induced maps $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $f_1 : H_1(X) \to H_1(Y)$. Prove commutativity of the diagram

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "f_*"] \arrow[d, "\rho_X"] & \pi_1(Y,y_0) \arrow[d, "\rho_Y"] \\ H_1(X) \arrow[r, "f_1"] & H_1(Y) \end{tikzcd}$$

where ρ_X and ρ_Y are the homomorphisms defined in Exercise 2.

Solution: Let $\gamma : I \to X$ be a loop based at x_0. To avoid confusion we denote here the classes determined in the fundamental group and in homology respectively by $[\gamma] \in \pi_1(X, x_0)$ and $[[\gamma]] \in H_1(X)$. It follows straight from the definitions of f_*, f_1 and the Hurewicz homomorphisms ρ_X and ρ_Y that

$$f_1(\rho_X([\gamma])) = f_1[[\gamma]] = [[f \circ \gamma]] = \rho_Y([f \circ \gamma]) = \rho_Y(f_*[\gamma]).$$

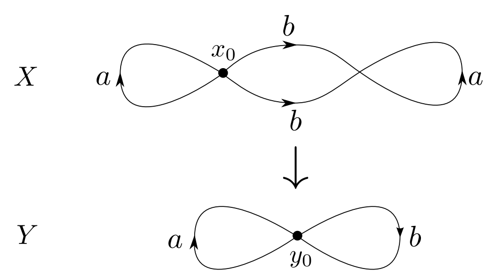
Since this works for every γ, we conclude $f_1 \circ \rho_X = \rho_Y \circ f_*$.

4. Let $p : X \to Y$ be a covering map, and let $x_0 \in X$ and $y_0 = p(x_0)$. We have proven in class that $p_* : \pi_1(X, x) \to \pi_1(Y, y)$ is injective. Is it true in general that $p_* : H_1(X) \to H_1(Y)$ is injective?

   Solution:

   It is not true that $p_1 : H_1(X) \to H_1(Y)$ needs to be injective. For example, take any space Y with $H_1(Y) \neq 0$, set $X = Y \sqcup Y$, and consider the obvious double cover $p : X \to Y$. The induced map $p_* : H_1(X) \cong H_1(Y) \oplus H_1(Y) \to H_1(Y)$, $(\alpha, \beta) \mapsto \alpha + \beta$, is clearly not injective.

   For a slightly more involved example, consider $X = S^1 \vee S^1 \vee S^1$, $Y = S^1 \vee S^1$ and the covering map $p : X \to Y$ indicated by the following picture:



   Consider now the loop γ in X that starts at x_0 and then winds once around all of X in clockwise direction. This loop defines a non-zero element $[[\gamma]] \in H_1(X)$; but note that

$$p_1[[\gamma]] = \rho_Y(p_*[\gamma]) = \rho_Y[b^{-1}a^{-1}ba] = 0 \in H_1(Y),$$

   because $[b^{-1}a^{-1}ba]$ lies in the commutator of $\pi_1(Y, y_0)$, which is the kernel of the Hurewicz homomorphism ρ_Y. Thus $p_1 : H_1(X) \to H_1(Y)$ is not injective.

5. (a) Let X be a Δ complex. Show that $\chi(X) = \sum(-1)^i \dim \mathrm{C}_i^{\Delta}(X, \mathbb{R})$.
Solution: For every k we have

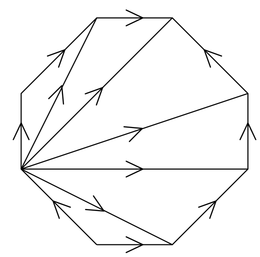
$$\dim C_k^{\Delta}(X, \mathbb{R}) = \dim \ker \partial_k + \dim \mathrm{Im} \partial_k.$$

In particular this implies

$\sum_{k=0}^{n}(-1)^k \dim C_k^{\Delta}(X, \mathbb{R}) =$
$(-1)^n \dim \ker \partial_n + \sum_{k=0}^{n-1}\left((-1)^k \dim \ker \partial_k - \dim \mathrm{Im}\partial_{k+1}\right) + \dim \mathrm{Im}\partial_0 =$
$(-1)^n \dim H_n^{\Delta}(X) + \sum_{k=0}^{n-1}(-1)^k \dim H_k^{\Delta}(X) + \dim \mathrm{Im}\partial_0 =$
$\sum_{k=0}^{n}(-1)^k \dim H_k^{\Delta}(X) + 0.$

(b) Compute the Euler characteristic of the surface of genus 2 (that is obtained from the regular octagon identifying parallel sides) and of the three dimensional torus $S^1 \times S^1 \times S^1$.

Solution: The surface of genus 2 has a Δ complex structure consisting of 1 0-simplex, 9 1-simplices, 6 2-simplices, in particular its Euler characteristic is -2, from the first part of the Exercise.



In order to give an easy Δ-complex structure on the three torus we first observe that the the product $[0,1]^3$ is the cone onto its boundary, in particular it can be written as the union of 6 pyramids with square basis. Each of those can be written as the union of two 3 simplices, and this subdivision can be made in a way that is coherent with the identifications in the three torus. In particular in this structure, we have 12 3 simplices, 24 2 simplices (each 2 dimensional faces of the Δ-complex structure is face of precisely 2 simplices, hence we have twice as many 2-simplices as 3-simplices), 14 1-simplices (8 in the interior, 3 subdividing the external squares, 3 edges of the square, after the identification) and 2 0-faces. In total the characteristic is zero.