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Algebraic Topology I
HS 2014

Solution 7

Computations in Homology

1. A pair of topological spaces (X, A) is a *good pair* if A is a nonempty closed subspace and there exists an open neighborhood V of A in X that deformation retracts onto A. Prove that, if (X, A) is a good pair, it holds $H_n(X, A) = H_n(X/A)$ for each $n > 0$.

   Solution: see Hatcher Proposition 2.22. In the last step you need to notice that if $x \in X$ is a point, then $H_n(X, \{x\}) \cong H_n(X)$ for each $n > 0$.

2. Compute the homology of the projective spaces $\mathbb{P}^n(\mathbb{R})$:

   (a) Find a subspace of $\mathbb{P}^n(\mathbb{R})$ homeomorphic to $\mathbb{P}^{n-1}(\mathbb{R})$.

   Solution: the projective space $\mathbb{P}^n(\mathbb{R})$ is defined to be the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$. The quotient of the subspace

   $$(\mathbb{R}^n \times \{0\}) \setminus \{(0, \dots, 0)\}$$

   is a subspace of $\mathbb{P}^n(\mathbb{R})$ that is homeomorphic to $\mathbb{P}^{n-1}(\mathbb{R})$. It consists precisely of those points whose homogeneous cohordinates have the form $[x_0 : \dots : x_{n-1} : 0]$.

   (b) Show that the pair $(\mathbb{P}^n(\mathbb{R}), \mathbb{P}^{n-1}(\mathbb{R}))$ is a good pair.

   Solution: let us consider the neighborhood V of $\mathbb{P}^{n-1}$ consisting of $V = \mathbb{P}^n(\mathbb{R}) \setminus \{[0 : \dots : 0 : 1]\}$. It is easy to check that the map

   $$\begin{array}{rccc} r: & V \times [0,1] & \to & \mathbb{P}^{n-1}(\mathbb{R}) \\ & ([x_0 : \dots : x_n], t) & \mapsto & [x_0 : \dots : (1-t)x_n] \end{array}$$

   is well defined and gives the desired retraction.

(c) Determine the homotopy type of $\mathbb{P}^n(\mathbb{R})/\mathbb{P}^{n-1}(\mathbb{R})$.

Solution: It is easy to check that $\mathbb{P}^n(\mathbb{R})/\mathbb{P}^{n-1}(\mathbb{R})$ is homeomorphic to an n-dimensional sphere: indeed $\mathbb{P}^n(\mathbb{R})$ is also the quotient of the upper hemisphere in the n-dimensional sphere under the antipodal map, and $\mathbb{P}^{n-1}(\mathbb{R})$ is the quotient of the equator. Once the equator is collapsed to a point, we get that the $\mathbb{P}^n(\mathbb{R})/_{\mathbb{P}^{n-1}(\mathbb{R})}$ is a CW complex with a single n-dimensional cell attached to a single 0-cell.

(d) Compute $H_k(\mathbb{P}^2(\mathbb{R}))$.

Solution: If $n = 1$ then $\mathbb{P}^1(\mathbb{R}) \cong S^1$ and hence

$$H_k(\mathbb{P}^1(\mathbb{R})) = \begin{cases} \mathbb{Z} \text{ if } k = 0, 1 \\ 0 \text{ otherwise} \end{cases}$$

The long exact sequence of the pair gives us

$$0 = H_2(\mathbb{P}^1(\mathbb{R})) \longrightarrow H_2(\mathbb{P}^2(\mathbb{R}))) \longrightarrow H_2(S^2) \xrightarrow{\partial}$$

$$H_1(\mathbb{P}^1(\mathbb{R})) \longrightarrow H_1(\mathbb{P}^2(\mathbb{R})) \longrightarrow H_1(S^2) \longrightarrow 0$$

In particular since the connecting morphism is given by the multiplication by 2, we get that $H_2(\mathbb{P}^2(\mathbb{R})) = 0, H_1(\mathbb{P}^2(\mathbb{R})) = \mathbb{Z}/_{2\mathbb{Z}}$, and $H_0(\mathbb{P}^2(\mathbb{R})) = \mathbb{Z}$.

(e) Prove, by induction, that $H_k(\mathbb{P}^n(\mathbb{R}))$ is zero if k is even and bigger than zero, is $\mathbb{Z}/_{2\mathbb{Z}}$ if k is odd and smaller than n, and is $\mathbb{Z}$ if $k = n$ are odd.

Solution: Let us consider again the long exact sequence of the pair $(\mathbb{P}^n(\mathbb{R}), \mathbb{P}^{n-1}(\mathbb{R}))$: this gives us

$$\ldots \longrightarrow H_k(\mathbb{P}^{n-1}(\mathbb{R})) \longrightarrow H_k(\mathbb{P}^n(\mathbb{R}))) \longrightarrow H_k(S^n) \longrightarrow \ldots$$

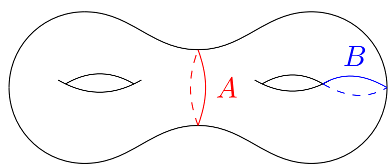
Since $H_k(S^n) = 0$ unless $k = n, 0$ we have that $H_k(\mathbb{P}^n(\mathbb{R})) \cong H_k(\mathbb{P}^n(\mathbb{R}))$ for $k < n-1$, in particular we just need to argue for $k = n, n-1$. In that case we have

$$0 = H_n(\mathbb{P}^{n-1}(\mathbb{R})) \longrightarrow H_n(\mathbb{P}^n(\mathbb{R}))) \longrightarrow H_n(S^n) \xrightarrow{\partial}$$

$$H_{n-1}(\mathbb{P}^{n-1}(\mathbb{R})) \longrightarrow H_{n-1}(\mathbb{P}^n(\mathbb{R})) \longrightarrow H_{n-1}(S^n) \longrightarrow$$

If n is odd we know, by induction, that $H_{n-1}(\mathbb{P}^{n-1}(\mathbb{R}))$ is zero hence in particular $H_n(\mathbb{P}^n(\mathbb{R})) \cong H_n(S^n) \cong \mathbb{Z}$ and $H_{n-1}(\mathbb{P}^n(\mathbb{R})) = H_{n-1}(\mathbb{P}^{n-1}(\mathbb{R})) = 0$. If instead n is even, we have that the connecting morphism $\partial : H_{2s}(S^{2s}) \to H_{2s-1}(\mathbb{P}^{2s-1}(\mathbb{R}))$ is induced by the multiplication by 2. In particular this tells us that $H_{2s}(\mathbb{P}^{2s}(\mathbb{R})) = 0$, since the multiplication by 2 gives an injective map, and $H_{2s-1}(\mathbb{P}^{2s-1}(\mathbb{R})) = \mathbb{Z}/_{2\mathbb{Z}}$.

3. Compute the relative homology $H_k(\Sigma_2, A)$ and $H_k(\Sigma_2, B)$ where Σ_2 is the surface of genus 2 and A (resp. B) is the curve depicted in red (resp. blue).



Solution: recall that for the surface of genus two it holds $H_2(\Sigma_2) = \mathbb{Z}$, $H_1(\Sigma_2) = \mathbb{Z}^4$, $H_0(\Sigma_2) = \mathbb{Z}$: this can be computed either directly using the simplicial homology, or realizing the surface as the quotient of the octagon by identifying parallel sides and considering the good pair given by (Σ_2, X) where $X \subset \Sigma_2$ is the graph that is quotiente of the boundary of the octagon.

In particular for the long exact sequence of the pair we have

$$\begin{array}{cccccccc} 0 \to & H_2(\Sigma_2) \to & H_2(\Sigma_2, X) \to & H_1(X) \to & H_1(\Sigma_2) \to & H_1(\Sigma_2, X) \to & 0 \\ & \mathbb{Z} & & \mathbb{Z} & \mathbb{Z}^4 & & \end{array}$$

In the first case we have that the map $H_1(A) \to H_1(\Sigma_2)$ is the zero map, hence we get $H_1(\Sigma_2, X) \cong H_1(\Sigma_2) \cong \mathbb{Z}^4$ and $H_2(\Sigma_2, A) \cong \mathbb{Z}^2$. In the second case we have that the map $H_1(A) \to H_1(\Sigma_2)$ is the inclusion of one of the generators, in particular we have $H_2(\Sigma_2, B) \cong H_2(\Sigma_2) \cong \mathbb{Z}$ and $H_1(\Sigma_2, A) \cong \mathbb{Z}^3$.

Notice that the topological space Σ_2/A is homeomorphic to $T^2 \vee T^2$ where T^2 is the two dimensional torus, and Σ_2/B is homotopic equivalent to $T^2 \vee S^1$ being homomorphic to a torus with two points identified.

4. Let X, Y be CW complexes, compute the homology of $X \vee Y$ assuming that the homology of X and Y are known.

   Solution: Since X and Y are CW complexes, we have that the space $X \vee Y$ can be endowed with a CW complex structure containing the wedge point as a zero cell. In particular the pair $(X \vee Y, X)$ is a good pair and we can apply the long exact sequence of the pair to compute the homology of $X \vee Y$. It is easy to check, using the explicit description for the connection morphism that in this case the connecting morphism is always the trivial map, and in particular we have short exact sequences

   $$0 \to H_k(X) \to H_k(X \vee Y) \to H_k(Y) \to 0.$$

   Moreover all these sequences split since there exists a retraction $r : X \vee Y \to X$ that induces maps $r_k : H_k(X \vee Y) \to H_k(X)$ satisfying $r_k \circ i_k = \mathrm{id}_{H_k(X)}$.

5. Consider the spaces $X = S^2 \vee S^1 \vee S^1$ and $Y = S^1 \times S^1$. Show that $H_k(X) = H_k(Y)$ for all k, but there exists no homotopy equivalence between X and Y.

   Solution: For Z equal to either X or Y we have $H_0(Z) = \mathbb{Z}, H_1(Z) = \mathbb{Z}^2, H_2(Z) = \mathbb{Z}$. This is a consequence of the fact that the singular and simplicial homology are equal and of the computation of Problem 4 in Sheet 4 for the space Y, and is a consequence of Exercise 3 for the space $X = S^2 \vee S^1 \vee S^1$. An easy way to show that there exists no homotopy equivalence between the two spaces is by checking that the fundamental groups of the spaces are not isomorphic, since $\pi_1(X, x) = \mathbb{F}_2$ and $\pi_1(Y, y) = \mathbb{Z}^2$.

6. (a) Show that the quotient map $S^1 \times S^1 \to S^2$ obtained collapsing the subspace $S^1 \vee S^1$ to a point is not null-homotopic.

   Solution: The quotient map cannot be null-homotopic, since both $H_2(S^1 \times S^1)$ and $H_2(S^2)$ are isomorphic to $\mathbb{Z}$ and the map induced from the quotient map is an isomorphism. This last fact can be checked using the exact sequence of the pair and noticing that the connecting morphism $\partial : H_2(S^2) \to H_1(S^1 \vee S^1)$ is the zero map.

(b) Show that any map $S^2 \to S^1 \times S^1$ is null-homotopic.

Solution: since S^2 is simply connected, any map $f : S^2 \to S^1 \times S^1$ lifts to a map $\overline{f}$ with values in the universal cover $\mathbb{R}^2$ of $S^1 \times S^1$. In particular, since $\mathbb{R}^2$ is contractible, the map $\overline{f}$ is null homotopic.