Exercise sheet 11

The content of the marked exercises **(*)** should be known for the exam.

1. For the following values of $\alpha \in \mathbb{C}$, find the minimal polynomial of α over $\mathbb{Q}$:

- $\alpha = \sqrt{2} + \sqrt{5}$
- $\alpha = \sqrt{3} - \sqrt[3]{3}$
- $\alpha = \lambda + i\lambda$, where $\lambda \in \mathbb{R}_{>0}$, $\lambda^4 = 5$.

2. Suppose that the field extension $L = K(\alpha)$ over K is finite of odd degree. Prove: $L = K(\alpha^2)$.

3. (*) (Trace and norm for finite field extensions) Let L over K be a finite field extension.

1. For $x \in L$, show that the following is a K-linear map:

$$\begin{aligned} m_x : L &\to L \\ y &\mapsto xy. \end{aligned}$$

When $K = \mathbb{R}$, $L = \mathbb{C}$ and $\alpha \in \mathbb{C}$, compute the matrix representing m_α with respect to the basis $(1, i)$.

2. Show that we have an injective ring homomorphism

$$\begin{aligned} r_{L/K} : L &\to \mathrm{End}_K(L) \\ x &\mapsto m_x. \end{aligned}$$

3. Consider the maps

$$\begin{aligned} \mathrm{Tr}_{L/K} : L &\to K \qquad \text{(trace map)} \\ x &\mapsto \mathrm{Tr}(m_x) \end{aligned}$$

and

$$\begin{aligned} \mathrm{N}_{L/K} : L &\to K \qquad \text{(norm map)} \\ x &\mapsto \det(m_x). \end{aligned}$$

Prove:

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- $\mathrm{Tr}_{L/K}$ is K-linear
- $\mathrm{N}_{L/K}(xy) = \mathrm{N}_{L/K}(x)\mathrm{N}_{L/K}(y)$ for every $x, y \in L$, and $\mathrm{N}_{L/K}(x) = 0$ if and only if $x = 0$.

4. Given a tower of finite extensions $L_1/L_2/K$, show that

$$\mathrm{Tr}_{L_1/K} = \mathrm{Tr}_{L_2/K} \circ \mathrm{Tr}_{L_1/L_2} .$$

[*Hint:* Get a K-basis for L_1 starting from a K-basis for L_2 and an L_2-basis for L_1, then evaluate the right hand side on $\alpha \in L_1$].

5. Prove that if $x \in L$ is such that $L = K(x)$, and

$$\mathrm{Irr}(x, K)(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0 \in K[X],$$

then $\mathrm{Tr}_{L/K}(x) = -a_{d-1}$ and $\mathrm{N}_{L/K}(x) = (-1)^d a_0$. [*Hint:* $(1, x, \ldots, x^{d-1})$ is a K-basis of L.]

6. Let p be an odd prime number, $\zeta_p = e^{\frac{2\pi i}{p}}$ and $K_p = \mathbb{Q}(\zeta_p)$. Find $\mathrm{Irr}(\zeta_p, \mathbb{Q})$, $\mathrm{Tr}_{K_p/\mathbb{Q}}(\zeta_p)$, $\mathrm{N}_{K_p/\mathbb{Q}}(\zeta_p)$ and $\mathrm{N}_{K_p/\mathbb{Q}}(\zeta_p - 1)$. [*Hint:* Look at Exercise 2.4 from Exercise sheet 9. Use previous point, and notice that $\mathbb{Q}(\zeta_p) = \mathbb{Q}(\zeta_p - 1)$.]

4. Prove that for every algebraic field extension $K/\mathbb{R}$ we have that K is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.

Due to: 4 December 2014, 3 pm.