Exercise sheet 3

The content of the marked exercise **(*)** should be known for the exam.

1. For a fixed integer $n \geq 3$, Consider $H = \{\sigma \in S_n | \sigma(1) = 1, \sigma(n) = n\} \subseteq S_n$, and $X_{i,j} := \{\sigma \in S_n | \sigma(1) = i, \sigma(n) = j\} \subseteq S_n$, for $i \neq j$ and $1 \leq i, j \leq n$.

1. Prove that H is a subgroup of S_n, whose right cosets σH are precisely the $X_{i,j}$.
2. For $\sigma \in S_n$, give a necessary and sufficient condition for $\sigma \in X_{i,j}$.
3. For $n \geq 4$, show that H is not a normal subgroup of S_n. What happens for $n = 3$?

2. Let G be a group, and $H \leq G$ be a subgroup.

1. Prove that the inversion map $i : G \to G$ induces a well-defined map $G/H \to H\backslash G$.
2. Show that $|H\backslash G| = |G/H|$. When it is finite, we denote this cardinality by $[G : H]$, and call it the *index* of H in G.
3. Now suppose that K is an intermediate subgroup, $H \leq K \leq G$, and that H has finite index in G. Prove that $[G : H] = [G : K][K : H]$ and in particular H has finite index in K and K has finite index in G. [*Hint*: Decompose G into cosets of H starting from decompositions of G into cosets of K and of K into cosets of H.]

3. Given two groups H and K, we define on the cartesian product $H \times K$ the operation $(h, k) \cdot (h', k') = (hh', kk')$.

1. Show that $(H \times K, \cdot)$ is a group. It is called the *direct product* of the groups H and K.
2. Consider the subsets $H' = H \times \{1_K\}$ and $K' = \{1_H\} \times K$ of $H \times K$. Prove that they are normal subgroups of $H \times K$, and that each element $x \in H \times K$ can be written in an unique way as a product $x = h'k'$, with $h' \in H'$ and $k' \in K'$.
3. Now suppose that G is a group, and suppose that H and K are normal subgroups of G such that $H \cap K = \{1_G\}$ and $HK = \{hk \in G | h \in H, k \in K\} = G$. Prove: G is isomorphic to $H \times K$.

4. Let G be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of all matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ for $a, b \in \mathbb{R}$ (and $a \neq 0$). Find a normal subgroup H of G such that G/H is isomorphic to $\mathbb{R}^\times$, and write down an explicit isomorphism.

Please turn over!

5. Let G be a group. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

6. Let G be a group. Define $Z(G) = \{x \in G \mid xz = zx \text{ for all } z \in G\}$. Show that $Z(G)$ is a normal subgroup of G (it is called the *center* of G).

7. **(*)** Let G be a group. For $a, b \in G$, we define the *commutator of a and b* via $[a, b] := aba^{-1}b^{-1}$. Moreover, consider the *commutator subgroup* $[G, G] = \langle [a, b] \rangle_{a,b \in G}$.

1. Show that the inverse of a commutator is a commutator, and that $[a, b] = 1_G$ if and only if a and b commute.
2. Show that $[G, G]$ is a normal subgroup of G.
3. Show that the quotient group $G/[G, G]$ is abelian. We denote it by G^{ab}, and call it the *abelianization of* G.
4. Suppose that A is an abelian group, and that $\phi : G \to A$ is a group homomorphism. Prove: $[G, G] \subseteq \ker(\phi)$.
5. Deduce that for each $\phi : G \to A$ as above there exists a unique group homomorphism $\psi : G^{\mathrm{ab}} \to A$ such that $\phi = \psi \circ \pi$, where π is the natural projection $G \to G^{\mathrm{ab}}$.

Due to: 9 October 2014, 3 pm.