Solutions of exercise sheet 3

The content of the marked exercise **(*)** should be known for the exam.

Recall that for G a group and H a subgroup of G we call a left (resp., right) coset of H any subset of G of the form Hg (resp., gH), with $g \in G$. We denote

$$G/H = \{gH | g \in G\} \quad \text{and} \quad H\backslash G = \{Hg | g \in G\}$$

If H is a normal subgroup, we denote with G/H the quotient group.

1. For a fixed integer $n \geq 3$, Consider $H = \{\sigma \in S_n | \sigma(1) = 1, \sigma(n) = n\} \subseteq S_n$, and $X_{i,j} := \{\sigma \in S_n | \sigma(1) = i, \sigma(n) = j\} \subseteq S_n$, for $i \neq j$ and $1 \leq i, j \leq n$.

1. Prove that H is a subgroup of S_n, whose right cosets σH are precisely the $X_{i,j}$.
2. For $\sigma \in S_n$, give a necessary and sufficient condition for $\sigma \in X_{i,j}$.
3. For $n \geq 4$, show that H is not a normal subgroup of S_n. What happens for $n = 3$?

Solution:

1. It is immediate to check that H is a subgroup: the identity fixes 1 and n, and if $\sigma, \tau \in H$, it is clear that both σ^{-1} and $\sigma\tau$ fix 1 and n.
   Now take two couple of indexes $i \neq j$ and $i' \neq j'$, and suppose that $\sigma \in X_{i,j}$ and $\sigma' \in X_{i',j'}$. Then one has
   $$\tau \in \sigma H \iff \sigma^{-1}\tau \in H \iff \sigma^{-1}\tau(1) = 1 \;\&\; \sigma^{-1}\tau(n) = n \iff$$
   $$\iff \tau(1) = \sigma(1) \,\&\, \tau(n) = \sigma(n) \iff \tau \in X_{\sigma(1),\sigma(n)}$$
   so that $\sigma H = X_{\sigma(1),\sigma(n)}$ and the $X_{i,j}$ are all the right cosets of H.
2. For $i \neq j$ indexes, denote by $\tau_{i,j}$ the permutation which transpose 1 with i and n with j, and fixes all the non-mentioned elements. Then $\tau_{i,j}(1) = i$ and $\tau_{i,j}(n) = j$ by definition, so that $X_{i,j} = \tau_{i,j}H$ by the previous point. Then we have the following equivalent conditions:
   $$\sigma \in X_{i,j} \iff \sigma \in \tau_{i,j}H \iff \tau_{i,j}^{-1}\sigma \in H \iff \exists h \in H : \sigma = \tau_{i,j} \circ h$$
3. Suppose that $n \geq 4$. Then $1 < 2 < 3 < n$. Call $\tau = (2\ 3)$ and $\sigma = (1\ 2)$. Clearly, $\tau \in H$. But $\sigma\tau\sigma^{-1} = (1\ 2)(2\ 3)(1\ 2) = (1\ 3) \notin H$, so that $\sigma H \sigma^{-1} \not\subseteq H$ and H is not normal.
   If $n = 3$, any permutation $\sigma \in H$ is forced to fix 2 as well, so that $H = 1$, which is clearly a normal subgroup, since it is stable under conjugation.

Please turn over!

2. Let G be a group, and $H \leq G$ be a subgroup.

1. Prove that the inversion map $i : G \to G$ induces a well-defined map $G/H \to H\backslash G$.
2. Show that $|H\backslash G| = |G/H|$. When it is finite, we denote this cardinality by $[G : H]$, and call it the *index* of H in G.
3. Now suppose that K is an intermediate subgroup, $H \leq K \leq G$, and that H has finite index in G. Prove that $[G : H] = [G : K][K : H]$ and in particular H has finite index in K and K has finite index in G. [*Hint*: Decompose G into cosets of H starting from decompositions of G into cosets of K and of K into cosets of H.]

Solution:

1. A map $\bar{i} : G/H \to H\backslash G$ induced by i should be defined by $gH \mapsto Hg^{-1}$. We prove that this is a well-defined map. We have
$$gH = g'H \iff g^{-1}g' \in H \iff Hg^{-1}g' = H \iff Hg^{-1} = Hg'^{-1},$$
which proves not only that $\bar{i}$ is well-defined (by considering the implication $\Rightarrow$), but also that it is injective (by considering the converse implication).
2. The map $\bar{i}$ from the previous point is surjective (since for each $g \in G$ we have $\bar{i}(g^{-1}H) = Hg$), and being it also injective as we saw, it is a bijection. Hence $|G/H| = |H\backslash G|$.
3. Since cosets form a partition of a group, we can choose (using AC) sets of indexes I and J and families of elements $(g_i)_{i\in I}$ and $(k_j)_{j\in J}$, with $g_i \in G$ and $k_j \in K$ such that we have *disjoint* unions
$$G = \bigcup_{i\in I} g_i K \text{ and } K = \bigcup_{j\in J} k_j H.$$
Notice that $|I| = [G : K]$ and $|J| = [K : H]$. Then
$$G = \bigcup_{i\in I} g_i \bigcup_{j\in J} k_j H = \bigcup_{(i,j)\in I\times J} g_i k_j H \quad (**),$$
and if we prove that this union is disjoint, then we will get $|I \times J| = [G : H]$. Suppose that $g_i k_j H \cap g_{i'} k_{j'} H$ is non-empty, then there exist $h, h' \in H$ such that $x = g_i k_j h = g_{i'} k_{j'} h'$. Since $k_j h, k_{j'} h' \in K$, we have $x \in g_i K \cap g_{i'} K \neq \varnothing$, so that $i = i'$ by construction. Then $g_i k_j h = g_i k_{j'} h'$ gives $k_j h = k_{j'} h'$, and since this other element $g_i^{-1} x$ lies in $k_j H \cap k_{j'} H$, we also get $j = j'$. This proves that the union $(**)$ is disjoint. Then $|I \times J| = [G : H] < \infty$ by hypothesis, and since I and J are not empty, set theory gives $|I|, |J| < \infty$, and
$$[G : H] = |I \times J| = |I| \cdot |J| = [G : K][K : H].$$

3. Given two groups H and K, we define on the cartesian product $H \times K$ the operation $(h, k) \cdot (h', k') = (hh', kk')$.

See next page!

1. Show that $(H \times K, \cdot)$ is a group. It is called the *direct product* of the groups H and K.
2. Consider the subsets $H' = H \times \{1_K\}$ and $K' = \{1_H\} \times K$ of $H \times K$. Prove that they are normal subgroups of $H \times K$, and that each element $x \in H \times K$ can be written in an unique way as a product $x = h'k'$, with $h' \in H'$ and $k' \in K'$.
3. Now suppose that G is a group, and suppose that H and K are normal subgroups of G such that $H \cap K = \{1_G\}$ and $HK = \{hk \in G | h \in H, k \in K\} = G$. Prove: G is isomorphic to $H \times K$.

Solution:

1. We have that the operation $\cdot$ on $H \times K$ inherits associativity from associativity of the multiplications in H and K. The element $(1_G, 1_H)$ is readily checked to be neutral, and calling i_H and i_K the inversion maps of H and K, we obtain that $i_H \times i_K : H \times K \mapsto H \times K$, sending $(h, k) \mapsto (i_H(h), i_K(k))$ is an inversion map for $H \times K$. We can then conclude that $H \times K$ is a group with the operation $\cdot$ just defined.
2. It is immediate to check that the projection maps $\pi_1 : H \times K \to H$ and $\pi_2 : H \times K \to K$ are group homomorphisms. Then $H' = H \times \{1_K\} = \ker(\pi_2)$ and $K' = \{1_H\} \times K = \ker(\pi_1)$ are normal subgroups of $H \times K$.

   For $(h, k) \in H \times K, h' = (h_0, 1) \in H'$ and $k' = (1, k_0) \in K'$, it is clear that $(h, k) = h'k'$ if and only if $h = h_0$ and $k = k_0$, so that there exists a unique decomposition of the required form.
3. First, let us prove that the elements of H commute with the elements of K. Let $h \in H$ and $k \in K$. The equality $hk = kh$ is equivalent to the equality $hkh^{-1}k^{-1} = 1_G$. To show this, notice that $hkh^{-1} \in K$ (being K a normal subgroup) and $kh^{-1}k \in H$ (being H a normal subgroup), and that the left hand side lies in both H and K. Being the intersection of the two subgroups trivial, we get $hkh^{-1}k^{-1} = 1_G$. Hence elements in H commute with elements in K.

   Now let us define the following map, where the group structure on $H \times K$ is defined as before:

   $$\begin{aligned} \phi : H \times K &\to G \\ (h, k) &\mapsto hk \end{aligned}$$

   Then ϕ is a group homomorphism: for every $h, h' \in H$ and $k, k' \in K$ we have indeed

   $$\phi((h, k)(h', k')) = \phi((hh', kk')) = hh'kk' \stackrel{*}{=} hkh'k' = \phi(h, k) \cdot \phi(h', k'),$$

   where in passage $*$ we used the fact that elements of H commute with elements of K.

   Since $HK = G$ by hypothesis, we have that ϕ is surjective. As concerns the kernel, notice that if $hk = 1$ for $h \in H$ and $k \in K$, then one has $h = k^{-1} \in H \cap K = \{1_G\}$, so that the $\ker(\phi) = \{1_G\}$. In conclusion, ϕ establish an isomorphism of groups between G and $H \times K$ as desired.

Please turn over!

4. Let G be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of all matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ for $a, b \in \mathbb{R}$ (and $a \neq 0$). Find a normal subgroup H of G such that G/H is isomorphic to $\mathbb{R}^\times$, and write down an explicit isomorphism.

Solution:

G is easily checked to be a subgroup of $\mathrm{GL}_2(\mathbb{R})$. In particular we have the formula

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa' & ab' + b \\ 0 & 1 \end{pmatrix}, \ \forall 0 \neq a, b \in \mathbb{R}.$$

This formula implies that the projection of the first entry $\pi : G \to \mathbb{R}^\times$, sending $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto a$ is a group homomorphism. It is clearly surjective, so that $G/\ker(\pi)$ is isomorphic to $\mathbb{R}^\times$ by the first isomorphism theorem. Hence we get that G/H is isomorphic to $\mathbb{R}^\times$ for

$$H = \ker(\pi) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$$

More explicitly, the map π factors as $\pi = \bar{\pi} \circ p$,

$$G \xrightarrow{p} G/H \xrightarrow{\bar{\pi}} \mathbb{R}^\times$$

where $p : g \mapsto gH$ is the quotient map and $\bar{\pi}$ is the unique group isomorphism sending $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} H \mapsto a$.

5. Let G be a group. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

Solution:

Recall that $\mathrm{Inn}(G) = \{\sigma_c | c \in G\} \leq \mathrm{Aut}(G)$, where $\sigma_c(x) := cxc^{-1}$. To prove that this subgroup is normal, look at the conjugacy class in $\mathrm{Aut}(G)$ of any σ_c. For $\tau \in \mathrm{Aut}(G)$ and $c \in G$ we have

$$\forall x \in G, (\tau\sigma_c\tau^{-1})(x) = \tau(c\tau^{-1}(x)c^{-1}) = \tau(c)x\tau(c)^{-1},$$

so that $\tau\sigma_c\tau^{-1} = \sigma_{\tau(c)}$. This gives $\tau H \tau^{-1} \subseteq H$ for every $\tau \in \mathrm{Aut(G)}$, so that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

6. **(*)** Let G be a group. For $a, b \in G$, we define the *commutator of* a *and* b via $[a, b] := aba^{-1}b^{-1}$. Moreover, consider the *commutator subgroup* $[G, G] = \langle [a, b] \rangle_{a,b \in G}$.

1. Show that the inverse of a commutator is a commutator, and that $[a, b] = 1_G$ if and only if a and b commute.

See next page!

2. Show that $[G, G]$ is a normal subgroup of G.
3. Show that the quotient group $G/[G, G]$ is abelian. We denote it by G^{ab}, and call it the *abelianization of* G.
4. Suppose that A is an abelian group, and that $\phi : G \to A$ is a group homomorphism. Prove: $[G, G] \subseteq \ker(\phi)$.
5. Deduce that for each $\phi : G \to A$ as above there exists a unique group homomorphism $\psi : G^{\mathrm{ab}} \to A$ such that $\phi = \psi \circ \pi$, where π is the natural projection $G \to G^{\mathrm{ab}}$.

Solution:

1. For each $a, b \in G$ we have that $[a, b][b, a] = aba^{-1}b^{-1}bab^{-1}a^{-1} = 1$, so that the inverse of the commutator $[a, b]$ is $[b, a]$. Moreover, $[a, b] = 1_G$ if and only if $aba^{-1}b^{-1} = 1_G$, which is equivalent to $ab = ba$.
2. If H is a subgroup of G generated by a set S, one can prove that H is normal if and only if $gSg^{-1} \subseteq H$ for every $g \in G$, and check normality just on generators. But we will check normality directly on an element of the whole subgroup.

   Let $u \in H$ (*Watch out:* this does not mean that u is itself a commutator!), and $x \in G$. Then

   $$xux^{-1} = uu^{-1}xux^{-1} = u[u^{-1}, x] \in [G, G].$$

   In conclusion, $[G, G]$ is normal in G.
3. For every $x, y \in G$, one has $xy = [x, y]yx$. Hence the classes of x and y in the quotient group $G/[G, G]$ do commute.
4. For each $a, b \in G$ we obtain

   $$\phi([a, b]) = \phi(a)\phi(b)\phi(a)^{-1}\phi(b)^{-1} = \phi(aba^{-1}b^{-1})[\phi(a), \phi(b)] = 1_A$$

   since A is abelian. Then $[G, G] \subseteq \ker(\phi)$, since it is generated by elements in the kernel.
5. For each $g \in G$, we denote by $\bar{g} = \pi(g)$ the corresponding class in $G^{\mathrm{ab}} = G/[G, G]$. If a map $\psi : G^{\mathrm{ab}} \to A$ realises $\phi = \psi \circ \pi$, then $\psi(\pi(g)) = \phi(g)$, so that (being π surjective) there cannot exist two such maps which are different. To prove that $\psi(\bar{g}) = \phi(g)$ well defines a map ψ, suppose that $\bar{g}_1 = \bar{g}_2$. Then there exists $u \in [G, G]$ such that $g_1 = ug_2$, and this implies that $\phi(g_1) = \phi(ug_2) = \phi(u)\psi(g_2) = \phi(g_2)$ by applying the previous point. Hence ψ is a well-defined map, the unique one such that $\phi = \psi \circ \pi$. Finally, ψ is a group homomorphism:

   $$\psi(\bar{g}\bar{h}) = \phi(gh) = \phi(g)\phi(h) = \psi(\bar{g})\psi(\bar{h}).$$

7. Let G be a group. Define $Z(G) = \{x \in G \mid xz = zx \text{ for all } z \in G\}$. Show that $Z(G)$ is a normal subgroup of G (it is called the *center* of G).

Solution:

Please turn over!

We have a group homomorphism $\gamma : G \to \mathrm{Aut}(G)$ sending $g \mapsto \sigma_g$, where σ_g is the conjugation by g, as defined in the previous exercise. It is indeed immediate to check on elements $x \in G$ that $\sigma_g \sigma_h = \sigma_{gh}$ for every $g, h \in G$. Then we get

$$\ker(\gamma) = \{g \in G \,|\, \forall x \in X : gxg^{-1} = x\} = \{g \in G \,|\, \forall x \in X : gx = xg\} = Z(G),$$

so that the center of G is a normal of subgroup of G.

This can also be proved directly. First, notice that $1_G \in Z(G)$. Moreover, for $g, h \in Z(G)$ we have that for every $x \in X$ one gets $ghx = gxh = xgh$, so that $gh \in Z(G)$. Also g^{-1} lies in the center: $g^{-1}x = (x^{-1}g)^{-1} = (gx^{-1})^{-1} = xg^{-1}$. This proves that $Z(G) \leq G$. Moreover, for every $g \in Z(G)$ and $x \in G$ one has $xgx^{-1} = gxx^{-1} = g$, so that $xZ(G)x^{-1} \subseteq Z(G)$ for every $x \in G$ and $Z(G)$ is normal in G.