Exercise sheet 5

The content of the marked exercises **(*)** should be known for the exam.

1. Let G be a group, and consider the set of maps $C(G) = \{f : G \to \mathbb{C}\}$.

1. Prove that defining $(g \cdot f)(x) = f(xg)$, for every $g, x \in G$ and $f \in C(G)$ we obtain an action of G on $C(G)$. Is it faithful?
2. If $|G| \neq 1$, find a non-trivial invariant subset of $C(G)$.

2. Let G be a group and suppose there is an action of G on a set X. For $H \subseteq G$, define $X^H = \{x \in X | \forall h \in H, h \cdot x = x\}$. Prove: if $H \trianglelefteq G$, then the action of G on X induces an action of G/H on X^H.

3. Let G act transitively on a finite set X, with $|X| \geq 2$. Show that there exists at least one element of $g \in G$ such that g has no fixed point.

4. Let G be a group acting on X. Show that the stabilizers of two elements in the same orbit are conjugate. What happens if for $x \in X$ one has $\mathrm{Stab}_G(x) \trianglelefteq G$?

5. Consider the group $G = \mathrm{GL}_n(\mathbb{R})$, where n is a positive integer, and let H be the subgroup consisting of diagonal matrices.

1. Suppose that $g \in H$ has distinct eigenvalues. Compute $C_G(g)$. Try to generalize this for $g \in G$ a (non-necessarily diagonal) diagonalizable matrix with distinct eigenvalues.
2. Now suppose that $n = 2$. Compute $N_G(H)$ and show that $N_G(N_G(H)) = N_G(H)$.

6. Let G be a finite group. Prove that any subgroup of index equal to the smallest prime dividing $|G|$ is normal. [*Hint:* Consider an action of G on the coset space with respect to the subgroup, and find its kernel.]

Please turn over!

7. (*) We want to give a proof of Sylow theorems. Given a prime number p and a finite group G, we call p-subgroup of G any subgroup of order equal to a power of p. We call *p-Sylow subgroup* of a finite group G any subgroup of order equal to the maximal power of p dividing $|G|$. (For instance, if $G = S_4$, then a 2-Sylow subgroup of G is a subgroup of order 8, and the only 5-Sylow subgroup is $\{1_G\}$).

1. Let G be a finite group, and write $G = p^n h$, with p a prime number, and n, h positive integers such that p does not divide h. Consider the set $\mathcal{P} = \{I \subseteq G : |I| = p^n\}$:
   a) Prove that the following defines an action of G on $\mathcal{P}$:
   $$\forall g \in G, \forall I \in \mathcal{P}, g \cdot I := gI = \{gi | i \in I\};$$
   b) Prove that p does not divide $|\mathcal{P}|$, and deduce that there exists an orbit $\mathcal{O} \subseteq \mathcal{P}$ of the action above whose cardinality is not divisible by p. Deduce that $|\mathcal{O}|$ divides h;
   c) Prove that $\bigcup_{S \in \mathcal{O}} S = G$, and deduce from this that $|\mathcal{O}| \geq m$. Find the cardinality of $H = \mathrm{Stab}_G(S_0)$, for $S_0 \in \mathcal{O}$.

   Conclude: any finite group G has a p-Sylow subgroup (*First Sylow Theorem*).
2. *Second Sylow Theorem.* Let P be a p-Sylow subgroup of G and Q a p-subgroup of G.
   d) Prove that the following defines an action of Q on G/P:
   $$\forall q \in Q, \forall g \in G, q \cdot gP := (qg)P;$$
   e) Prove that the cardinality of any orbit is 1 or is divisible by p. Deduce that there is a fixed point $gP \in G/P$, and that P contains a conjugate of Q.

   Conclude: p-Sylow subgroups of G are conjugate in G (*Second Sylow Theorem*).
3. Let n_p be the number of p-Sylow subgroup of G, and P a p-Sylow subgroup of G.
   f) Prove that P acts on $X := \{Q \text{ } p\text{-Sylow in } G\}$ by conjugation;
   g) Prove that the action above has precisely one fixed point, and that p divides the size of the other orbits.

   Conclude: p divides $n_p - 1$, that is, $n_p \equiv 1 \pmod{p}$ (*Third Sylow Theorem*).

Due to: 23 October 2014, 3 pm.