Exercise sheet 6

The content of the marked exercises **(*)** should be known for the exam.

1. Consider the set

$$\mathbb{H} = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\} \subset M_2(\mathbb{C}).$$

Prove:

1. $\mathbb{H}$ is a subring of $M_2(\mathbb{C})$, the ring of matrices with entrywise sum and row-times-column multiplication;
2. $\mathbb{H}$ is a division ring (it is called the ring of Hamilton quaternions);
3. $\mathbb{H}$ is non-commutative;
4. $\mathbb{H}$ is a $\mathbb{R}$-vector space of dimension 4, and there exists an $\mathbb{R}$-basis $(\mathbb{1}, \mathbb{i}, \mathbb{j}, \mathbb{k})$ for $\mathbb{H}$ such that $\mathbb{i}^2 = \mathbb{j}^2 = \mathbb{k}^2 = -\mathbb{1}$, $\mathbb{i}\mathbb{j} = \mathbb{k} = -\mathbb{j}\mathbb{i}$, $\mathbb{j}\mathbb{k} = \mathbb{i} = -\mathbb{k}\mathbb{j}$ and $\mathbb{k}\mathbb{i} = \mathbb{j} = -\mathbb{i}\mathbb{k}$.

2. Let X be a set, A a commutative ring and $C(X, A) = \{f : X \to A\}$ be the ring of functions $X \to A$, with pointwise sum and multiplication. Given a subset $Y \subseteq X$, define

$$I_Y = \{f \in C(X, A) : f|_Y = 0\}.$$

Prove that I_Y is an ideal, and prove that it is principal by finding a generator.

3. We say that a ring A is simple if $A \neq 0$ and the only ideals of A are 0 and A. Prove: $M_n(\mathbb{R})$ is a simple ring for every $n \in \mathbb{Z}_{>0}$.

4. Let $A \neq 0$ be a ring. We say that $e \in A$ is *idempotent* if $e^2 = e$. We say that an idempotent element $e \in A$ is non-trivial if $e \notin \{0, 1\}$.

1. Prove: if $e \in A$ is idempotent, then $1 - e$ is also idempotent.
2. Suppose that $e \neq 1$ is idempotent in A. Prove: e is not a unit of A.
3. Prove that if $e \in A$ is idempotent, then $B = eAe = \{eae : a \in A\}$, endowed with sum and multiplications from A, is a ring with $0_B = 0_A$ and $1_B = e$.
4. Find a non-trivial idempotent element of $A = \mathbb{Z}/10\mathbb{Z}$.

5. (*) Consider the set

$$X = \{(f, \varepsilon) | \varepsilon \in \mathbb{R}_{>0}, f :]-\varepsilon, \varepsilon[\to \mathbb{C} \text{ continuous map}\}.$$

Define a relation $\sim$ on X via

$$(f_1, \varepsilon_1) \sim (f_2, \varepsilon_2) \iff (\ \exists \varepsilon \in \mathbb{R}_{>0},\ \varepsilon \leq \min(\varepsilon_1, \varepsilon_2) :\ f_1|_{]-\varepsilon,\varepsilon[} = f_2|_{]-\varepsilon,\varepsilon[}\ ).$$

1. Show that $\sim$ is an equivalence relation. We write $[(f, \varepsilon)]$ for the class of (f, ε), and $A = X/\sim$.
2. Show that the following are well defined maps $A \times A \to A$, i.e. binary operations on A:

$$[(f_1, \varepsilon_1)] + [(f_2, \varepsilon_2)] = [(f_1|_{]-\varepsilon,\varepsilon[} + f_2|_{]-\varepsilon,\varepsilon[}, \varepsilon)],$$
$$[(f_1, \varepsilon_1)] \cdot [(f_2, \varepsilon_2)] = [(f_1|_{]-\varepsilon,\varepsilon[} \cdot f_2|_{]-\varepsilon,\varepsilon[}, \varepsilon)],$$
$$\text{where } \varepsilon = \min(\varepsilon_1, \varepsilon_2),$$

   and that they define a ring structure on A with $0_A = [(0, 1)]$ and $1_A = [(1, 1)]$.
3. Show that $(f, \varepsilon) \mapsto f(0)$ defines a ring homomorphism $A \to \mathbb{C}$. Deduce that $I = \{[(f, \varepsilon)] | f(0) = 0\}$ is an ideal of A, and that $A/I \cong \mathbb{C}$.
4. Prove: $A^\times = A \setminus I$.

The ring A we defined is called the ring of "germs of continuous functions at 0".

Due to: 30 October 2014, 3 pm.