Exercise sheet 7

The content of the marked exercises **(*)** should be known for the exam.

1. **(*)** Let R be a ring. Similarly as for groups, given two R-linear maps $\alpha : L \to M$ and $\beta : M \to N$ we say that

$$L \overset{\alpha}{\to} M \overset{\beta}{\to} N$$

is an exact sequence of R-modules if $\mathrm{Im}(\alpha) = \ker(\beta)$, and given

$$(**) \qquad \cdots \longrightarrow M_{n-2} \overset{\alpha_{n-2}}{\longrightarrow} M_{n-1} \overset{\alpha_{n-1}}{\longrightarrow} M_n \overset{\alpha_n}{\longrightarrow} M_{n+1} \overset{\alpha_{n+1}}{\longrightarrow} M_{n+2} \longrightarrow \cdots$$

we say that $(**)$ is an exact sequence if $M_{i-1} \overset{\alpha_{i-1}}{\longrightarrow} M_i \overset{\alpha_i}{\longrightarrow} M_{i+1}$ is an exact sequence for every i. We call a short exact sequence of R-modules any exact sequence of R-modules of the form

$$0 \to L \overset{\alpha}{\to} M \overset{\beta}{\to} N \to 0.$$

1. Show that in the exact sequence above N is determined, up to isomorphism, by the map α.
2. Find a short exact sequence as above, with $R = \mathbb{Z}$, $L = \mathbb{Z}$, $M = \mathbb{Z} \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$, and $\alpha(n) = 2n + 0$.
3. Find a short exact sequence as above, with $R = \mathbb{R}$, $L = M \oplus M$ and $M \neq 0$.

2. Let R be a ring and M be an R-module. Let $N \leq M$ and $L \leq M$, meaning that N and L are R-submodules of M.

Show that $N \cap L \leq N$, and that $L \leq N + L \leq M$, and prove that there is an isomorphism $N/(N \cap L) \overset{\sim}{\longrightarrow} (N + L)/L$.

3. Let $R \neq 0$ be a commutative ring. We say that $a \in R$ is a zero-divisor if there exists $b \in R$ such that $b \neq 0$ and $ab = 0$. We say that $a \in R$ is regular if a is not a zero-divisor.

1. Prove that invertible elements in R are regular. Is the converse true?
2. Let $R_{\mathrm{reg}} = \{a \in R : a \text{ is regular}\}$. Prove that R_{reg} contains 1_R and that it is stable under multiplication. This is also phrased by saying that the R_{reg} is a multiplicative subset of R.
3. Let now M be an R-module. Define $M_{\mathrm{tor}} = \{m \in M | \exists r \in R_{\mathrm{reg}} : r \cdot m = 0_M\}$. Prove that M_{tor} is a submodule of M. It is called the *torsion submodule* of M.

Please turn over!

4. We say that a module N is torsion-free if $N_{\text{tor}} = 0$. Prove: for every R-module M, the module M/M_{tor} is torsion-free.
5. Find the torsion submodule of the $\mathbb{Z}$-module $M = \mathbb{R}/\mathbb{Z}$. What is M/M_{tor}?

4. (*) Let R be a commutative ring. If M and N are R-modules, we define $\text{Hom}_R(M, N)$ as the set of R-linear maps $M \to N$. It is easily seen to be an R-module by defining

$$(f+g)(m) = f(m)+g(m), \ \ (a\cdot f)(m) := a\cdot(f(m)), \ \ \forall f, g \in \text{Hom}_R(M, N), \ a \in R, \ m \in M.$$

1. Let N be an R-module. For every R-linear map $f : M_1 \to M_2$, define

$$f^* : \text{Hom}_R(M_2, N) \to \text{Hom}_R(M_1, N)$$
$$g \mapsto g \circ f.$$

   Prove that f^* is also an R-linear map, and that we have the following properties:
   - $(f_1 \circ f_2)^* = f_2^* \circ f_1^*$, for every couple of R-linear maps $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$;
   - $\text{id}_M^* = \text{id}_{\text{Hom}_R(M,N)}$ for every R-module M.
2. Define a natural map $\text{Hom}_R(M_1 \oplus M_2, N) \to \text{Hom}_R(M_1, N) \oplus \text{Hom}_R(M_2, N)$ and prove that it is an isomorphism of R-modules.
3. Prove that for any exact sequence of R-modules $A \to B \to C \to 0$, one has that the corresponding

$$0 \to \text{Hom}_R(C, N) \to \text{Hom}_R(B, N) \to \text{Hom}_R(A, N)$$

   is also an exact sequence of modules.
4. Let $A = \text{End}_{\mathbb{R}}(M)$, where M is a countably infinite dimensional $\mathbb{R}$-vector space (i.e., M has an $\mathbb{R}$-basis $\mathcal{B} = (e_i)_{i\in\mathbb{Z}_{>0}}$). Prove that A^2 is isomorphic to A as an $\mathbb{R}$-vector space.[*Hint:* First, prove that $M \cong M \oplus M$ as $\mathbb{R}$ vector space.] What happens if M is finite dimensional? (What if M is uncountably infinite dimensional?)

Due to: 6 November 2014, 3 pm.