Exercise sheet 8

The content of the marked exercises **(*)** should be known for the exam.

1. (*) [Formal construction of the polynomial ring] Let A be a commutative ring and consider the set

$$V = \{(a_i) \mid i \in \mathbb{Z}_{\geq 0}, a_i \in A, a_i = 0 \text{ for } i \text{ large enough}\}.$$

Endowing V with componentwise sum and with the scalar multiplication $a \cdot (a_i) = (a \cdot a_i)$, we have that V is an A-module. Define a multiplication

$$V \times V \to V$$
$$((a_i),(b_i)) \mapsto (a_i)\cdot(b_i) = (c_i), c_i = \sum_{\substack{j,k \geq 0 \\ j+k=i}} a_j b_k$$

1. Show that this product is well defined.
2. Show that $(1_A, 0_A, 0_A, \dots)$ is a neutral element for this product, and that the product is associative, commutative and distributive with respect to addition. This allows us to conclude that V is a ring.
3. Let $Y := (\alpha_i)$, with $\alpha_1 = 1_A$ and $\alpha_i = 0_A$ for $i \neq 1$. For $j \geq 0$, find the sequence of elements β_i for which $Y^j = (\beta_i)$. Deduce that $(Y^j)_{j\geq 0}$ is a basis of V as an A-module.
4. Let B be a commutative ring, $f_0 : A \to B$ a ring homomorphism and $b \in B$. Prove that there exists a unique ring homomorphism $f : V \to B$ sending $f(Y) = b$ and $f(a \cdot 1_V) = f_0(a)$ for each $a \in A$.
5. Let M be an A-module and $T : M \to M$ an A-linear map. Show that there exists a unique V-module structure $\cdot_V$ on M such that $Y \cdot_V m = T(m)$ and $(a \cdot 1_V) \cdot_V m = a \cdot_A m$. Moreover, show that if M is finitely generated as an A-module, then so it is as a V-module. Is the converse true?
6. Prove that V and $A[X]$ are isomorphic rings.

Please turn over!

2. Let A be a commutative ring

1. Show that there exists a unique A-linear map
$$D : A[X] \to A[X]$$
such that
$$D(X^i) = iX^{i-1}, \quad i \geq 1$$
$$D(1) = 0.$$
Is D a ring homomorphism?
2. Prove that for all $P, Q \in A[X]$ one has
$$D(PQ) = PD(Q) + QD(P)$$
3. (Factorization Theorem) Now let $A = K$ be a field, and $P \in K[X]$. Prove that for every $\alpha \in K$ one has $P(\alpha) = 0$ if and only if P is divisible by $X - \alpha$, that is, there is a polynomial $Q \in K[X]$ such that $P(X) = (X - \alpha)Q(X)$ [*Hint:* One implication is immediate. For the other, divide P by $X - \alpha$.]
4. We say that $\alpha \in K$ is a multiple root of $P \in K[X]$ if P is divisible by $(X - \alpha)^2$. Prove: α is a multiple root of P if and only if $P(\alpha) = D(P)(\alpha) = 0$.

3. Let A be an integral domain. Show that $A[X]^\times = A^\times$.

4. Let K be a field, and consider the ideal I generated by X and Y in $K[X, Y]$. Show:

1. I is not principal;
2. I is maximal.

Due to: 13 November 2014, 3 pm.