Weihnachts-Serie
(Problem Set 12)

1. Perturbation Theory. Let $\Omega \subseteq \mathbb{R}^n$ be bounded and connected, $2 < p < 2^*$ and

$$\tilde{E} \colon H_0^1(\Omega) \to \mathbb{R}$$
$$u \mapsto \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{p} \int_\Omega |u|^p \, dx - \int_\Omega fu \, dx.$$

Let furthermore $E(u) = \frac{1}{2}(\tilde{E}(u) + \tilde{E}(-u))$ be the symmetrised functional.

Let $\varphi_1, \varphi_2, \ldots$ be the L^2-orthonormal eigenfunctions of $-\Delta$ with eigenvalues $0 < \lambda_1 < \lambda_2 \leq \ldots$ and let

$$X_j = \mathrm{span}\{\varphi_1, \ldots, \varphi_j\}$$

and

$$\Gamma = \{h \in C^0(H_0^1; H_0^1) \mid h(u) = -h(-u) \text{ and } h(u) = u \text{ if } \max\{\tilde{E}(u), \tilde{E}(-u)\} < 0\},$$
$$\beta_j = \inf_{h \in \Gamma} \sup_{u \in X_j} E(h(u)).$$

(a) Assume we have $\beta_j < \beta_{j+1}$ for some j. Show that we can find a constant C_j such that if $\|f\|_2 < C_j$, then $\tilde{E}$ attains a critical value $> \beta_j$.

(b) Show that $\beta_j \to \infty$ for $j \to \infty$, i.e. carry out the details of the proof in the lecture.

(c) Show that given any $k \in \mathbb{N}$, there is a constant $\tilde{C}_k$ such that the equation

$$\begin{cases} -\Delta u = u|u|^{p-2} + f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least k solutions whenever $\|f\|_2 \leq \tilde{C}_k$.

Remark: In contrast to the Theorem shown in the lecture we may work with all subcritical exponents $2 < p < 2^*$ here.