Exercise Sheet 1

1. a) Prove that a closed curve γ in \mathbb{R}^n has

$$\int_{\gamma} |k| \, ds \geqslant \frac{\pi}{2}.$$

b*) ¹]Prove that a closed curve γ in \mathbb{R}^2 has

$$\int_{\gamma} |k| \, ds \ge 2\pi.$$

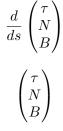
2. a) Compute the curvature k and torsion l at t = 0 for the curve

$$t \to (t, at^2, bt^3) \quad a, b \in \mathbb{R}$$

- b) Show that if a curve γ in \mathbb{R}^3 has identically vanishing torsion then γ lies in a plane.
- c) Suppose that a curve γ in \mathbb{R}^3 has a constant curvature and torsion. Show that γ must be a helix.
- d) Prove that any given smooth functions k(s), l(s), with k(s) > 0 determine a curve in \mathbb{R}^3 that is unique up to rigid motion of space (i.e. a composition of rotations and translations).

Hint: First compute

in terms of



$$\int_{\gamma} |k| \, ds \geqslant 2\pi.$$

Moreover there is an equality if and only if γ is a plane convex curve.

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¹For n = 3 we a similar result.

Theorem 0.1. Any closed curve γ in \mathbb{R}^3 has

Then apply the uniqueness and existence theorem for ODEs (you do not have to prove this!).

Theorem(*Existence and uniqueness for ODEs*²)

Let $U \subseteq \mathbb{R}^n$ be an open set. Let $f_1, \ldots f_n : U \to \mathbb{R}$ be Lipschitz continuous functions and let $x_0 := (x_0^1, x_0^2, \ldots, x_0^n)$ be a point of U. For any $t_0 \in \mathbb{R}$ consider the ODE system

$$(*) = \begin{cases} \dot{y}^{i}(t) = f^{i}(y^{1}(t), \dots, y^{n}(t)) & i = 1, \dots, n, \\ y^{i}(t_{0}) = x_{0}^{i} & i = 1, \dots, n. \end{cases}$$

then

- a) There exists a small open interval V containing t_0 and a continuously differentiable function $g(t) := (y^1(t), y^2(t), \dots, y^n(t)) : V \to \mathbb{R}^n$ that solves (*).
- b) Suppose that there are two solutions g, \tilde{g} of (*) defined on intervals V and \tilde{V} respectively. Then the two functions agree on the intersection $V \cap \tilde{V}$.
- 3. (For those new to topology)
 - a) Let X be a topological space. Show that if X is path connected, then X is connected.
 - b) Let X, Y be a topological spaces. Show that if X is connected and the map $f : X \to Y$ is continuous, then f(X) is connected.
 - c) Let $U \subseteq \mathbb{R}^n$ be open set. Show that if U is connected, then it is path connected.
- 4. a) Let $M \subseteq \mathbb{R}^3$ be a surface, $p \in M$, $X \in T_p M$ an unit vector. Show that

$$A_p(X,X) = \langle K_\gamma(p), N(p) \rangle,$$

where γ is any curve in M satisfying $\gamma(0) = p$ and $\frac{d\gamma}{ds}(0) = X$ and K_{γ} is its curvature vector.

b) Let M be a surface of revolution in \mathbb{R}^3 obtained by rotating a graph y = f(x) about the x-axis. Compute the principle curvatures and principle directions of M.

Due on Wednesday October 1(resp. Friday October 3)

 $^{^2}$ For a proof see for examples C. Blatter, Analysis 11.26 www.math.ethz.ch/~ blatter/