① Grassmannian — Exercise sheet 7

$$\mathcal{J}(\mathbb{R}^{2n}) = \{ J : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \mid \text{lin isometry (w.r.t Euclidean Metric), i.e } |Jv| = |v|, \; J^2 = -id \}$$

We say that a family of vector $v_1, \dots, v_n \in \mathbb{R}^{2n}$ is a complex basis if they define a complex basis of $\mathbb{C}^n$.
We consider

$$\mathcal{J}_0(\mathbb{R}^{2n}) = \{ J \in \mathcal{J}(\mathbb{R}^{2n}) \mid \text{let } e_1, \dots, e_n \text{ be a complex basis of } \mathbb{R}^{2n}, \text{ then } e_1, \dots, e_n, J(e_1), \dots, J(e_n) \text{ is a } \underline{\text{real}} \text{ basis of } \mathbb{R}^{2n} \text{ which has the same orientation of the standard basis of } \mathbb{R}^{2n} \}$$

Rmk: ① Let $e_1, \dots, e_n$ be a complex basis of $\mathbb{R}^{2n}$. Then

$$e_1, \dots, e_n, J(e_1), \dots, J(e_n)$$

is a real basis of $\mathbb{R}^{2n}$, for any $J \in \mathcal{J}(\mathbb{R}^{2n})$

Pf: Let $J \in \mathcal{J}(\mathbb{R}^{2n})$. J is an isometry, $J \in O(2n)$. Consider

$$\Phi : \mathbb{R}^{2n} \xrightarrow{\sim} \mathbb{C}^n \quad \text{linear map}$$
$$(x_1, y_1, \dots, x_n, y_n) \to (x_1 + iy_1, \dots, x_n + iy_n)$$

Suppose that $\exists\, w \in \operatorname{span}_{\mathbb{R}} \langle e_1, \dots, e_n \rangle$ s.t. $J(w) \in \operatorname{span}_{\mathbb{R}} \langle e_1, \dots, e_n \rangle$
Recall the block decomposition of $J \in O(n)$, it is equal to
$\exists\, V \in O(n)$ s.t.

$$VJV^T = \begin{pmatrix} R_{\theta_1} & & 0 \\ & \ddots & \\ 0 & & R_{\theta_n} \end{pmatrix} \qquad R_{\theta_j} = 2\times 2 \text{ matrix rotation of } \theta_j$$

Then if we change our coordinate we can consider $J_{\mathbb{R}^{2n}} = \begin{pmatrix} R_{\theta_1} & \cdots & 0 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R_{\theta_n} \end{pmatrix}$

Now if we consider $J_{\mathbb{R}^{2n}}$ in complex coordinate ($2n \times 2n$)

$$J_{\mathbb{C}^n} = \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix}$$

w.r.t $\underline{\Phi(e_1), \dots, \Phi(e_n)}$

$$J_{\mathbb{C}^n}(\Phi(w)) = \Phi\left(J_{\mathbb{R}^{2n}}(w)\right)$$

Thm: $w \in \mathrm{Span}_{\mathbb{R}} \langle e_1, \dots, e_n \rangle \Rightarrow \underline{\Phi}(w) \in \mathrm{Span}_{\mathbb{C}} \langle \underline{\Phi}(e_1), \dots, \underline{\Phi}(e_n) \rangle$

$w = \sum h_j e_j \quad , h_j \in \mathbb{R} \Rightarrow \underline{\Phi}(w) = \sum h_j \underline{\Phi}(e_j) \quad , h_j \in \mathbb{R}$

$J_{\mathbb{C}}\left(\underline{\Phi}(w)\right) = \sum h_j e^{i\theta_j} \underline{\Phi}(e_j)$

Since $J(w) \in \mathrm{Span}_{\mathbb{R}} \langle e_1, \dots, e_n \rangle$ we have that

$\exists\, \mu_j \in \mathbb{R} : 1 \quad J_{\mathbb{R}^{2n}}(w) = \sum_{j=1}^{n} \mu_j e_j$, then $\underline{\Phi}\left(J_{\mathbb{R}^{2n}}(w)\right) = \sum_{j=1}^{n} \mu_j \underline{\Phi}(e_j)$

Since $J_{\mathbb{C}} \underline{\Phi} = \underline{\Phi} J_{\mathbb{R}^{2n}}$ we have

$\mu_j = h_j \cdot e^{i\theta_j} \Rightarrow e^{i\theta_j} = \pm 1 \quad j = 1, \dots, n$

It follows that

$J_{\mathbb{R}^{2n}} = \begin{pmatrix} \pm 1 & \cdots & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix}$ — multiplication with ± 1 in $\mathbb{C}$ ⇕ $\underline{\Phi}$ — multiplication with $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ in $\mathbb{R}^2$

$2n \times 2n$ matrix

But $\left(J_{\mathbb{R}^{2n}}\right)^2 = \underline{1} \Rightarrow$ ↯ ☒

② Let $v_1, \dots, v_n$, $f_1, \dots, f_n$ be two complex basis of $\mathbb{R}^{2n}$. Then the change of basis matrix A defined by

$A(v_i) = f_i \quad i = 1, \dots, n$

$A(J(v_i)) = J(f_i) \quad i = 1, \dots, n$

has determinant > 0 [a fun algebra exercise]

$\Rightarrow$ Conclusion: $J_0(\mathbb{R}^n) \cup J_1(\mathbb{R}^{2n}) = J(\mathbb{R}^{2n})$ with the basis

Now: consider $S^3 \subset \mathbb{R}^4$ equipped with the quaternions, i.e. $v \in S^3$, $v = a \cdot 1 + bi + cj + dk$

$|v| = a^2 + b^2 + c^2 = 1$

Note: for $u \in Q$ $L_u : \underset{v}{Q} \longrightarrow \underset{uv}{Q}$, $R_u : \underset{v}{Q} \longrightarrow \underset{vu}{Q}$ are defined

Claim: $L_u, R_u \in J(\mathbb{R}^4)$ for $u \in S^3$

Pf: Let $v \in \mathbb{R}^4$, then $v = x + iy + jz + dk$, and

$L_u : \underset{v}{\mathbb{R}^4} \longrightarrow \underset{u.v}{\mathbb{R}^4}$

Thm • $L_u \circ L_u(v) = u \cdot u \cdot v = -u \cdot \bar{u} v = -v = -Id(v)$

$u \in S^2$, $u = ai + bj + ck$, $\bar{u} = -u$, $u^{-1} = \frac{\bar{u}}{|u|^2}$

• $|L_u(v)|^2 = uv\bar{u}v = uv\bar{v}\bar{u} = |v| \cdot |u| = |v|$

Analogously for R_u ∎

Claim $L_u \in \Gamma_+(\mathbb{R}^4)$, $R_u \in \Gamma_-(\mathbb{R}^4)$

pf The standard basis of $\mathbb{R}^4$ is $1, i, j, k$, then $u = ai + bj + ck$

$$\begin{aligned} 1 &\longrightarrow ai + bj + ck = u \cdot 1 \\ i &\longrightarrow -a - bk + cj = u \cdot i \\ j &\longrightarrow ak - b + ci = u \cdot j \\ k &\longrightarrow -aj + bi - c = u \cdot k \end{aligned}$$

Thm
$$A = \begin{pmatrix} 0 & -a & -b & -c \\ a & 0 & -c & b \\ b & c & 0 & -a \\ c & -b & a & 0 \end{pmatrix}$$

$$\begin{aligned} \det(A) &= a^2c^2 + b^4 - a^2c^2 - (a^4 + c^4) \\ &= b^4 - 2a^2c^2 - a^4 - c^4 \\ &= b^4 - (a^2 + c^2)^2 \\ &= b^4 - (1 - b^2)^2 \\ &= 1 + 2b^2 > 0 \end{aligned}$$

Analogously $R_u \in \Gamma_-(\mathbb{R}^4)$.

Thm We have a map $\Psi : S^2 \longrightarrow \Gamma_+(\mathbb{R}^4)$, $u \longmapsto L_u$

This map is inj,

Claim Ψ is surj

pf Let $\Gamma \in \Gamma_+(\mathbb{R}^4)$, then $\Gamma \in O(4)$ and since $\Gamma^2 = id$ we have $\Gamma^{-1} = \Gamma^T = -\Gamma$

Since $\Gamma(1) \in \mathbb{R}^4$, $\Gamma(1) = a + ib + jc + dk$.

Note that $v \in \mathbb{R}^4$,

$$\langle -\Gamma(v), v \rangle = \langle \Gamma^2(v), \Gamma(v) \rangle = \langle -v, \Gamma(v) \rangle = -\langle \Gamma(v), v \rangle = 0$$

Then $J(1)$ is orthogonal to $\mathbb{R}$ $J(i)$ is orthogonal to $\mathbb{R}i$,
$J(i)$ is " to $\mathbb{R}i$ $J(k)$ " " $\mathbb{R}k$

In particular $J(1) \in S^2$. Let $J(1) = ai+bj+ck$.
Note that

Let $V = \operatorname{span}(1, J(1))$, then $J: V \to V$ since $J(1) \in V$, $J^2(1) = -1 \in V$.

Note that $L_{J(1)} = J$ on V. Let $u = J(1)$

Now let $V^\perp = \operatorname{span}(1, J(1))^\perp$. Since J is orthogonal $\Rightarrow J(V^\perp) \subseteq V^\perp$

Since L_u is an isometry (i.e. orthogonal!), $L_u(V) \subset V$ it follows $L_u(V^\perp) \subseteq V^\perp$

Then $L_u : V^\perp \to V^\perp$ s.t. $L_u^2 = J^2 = -id$

$J : V^\perp \to V^\perp$

It follows that (by orthogonal matrix decomposition between 2×2 matrices)

$L_u = J \in \{R_{\frac{\pi}{2}}, R_{\frac{3\pi}{2}}\}$ R_θ = rot by θ

$\Rightarrow L_u = J$ or $L_u = J^{-1}$

Since $J \in J_+(\mathbb{R}^4)$ they are the same

Analogously $R : S^2 \to J_+(\mathbb{R}^4)$ is a bijection.

We conclude.

- $L : S^2 \to J_0(\mathbb{R}^4) \subseteq O(4)$ is a smooth, big map since it map has within as

$$ai+bj+ck \longrightarrow \begin{pmatrix} 0 & a & -b & c \\ -a & 0 & c & b \\ b & c & 0 & a \\ c & -b & a & 0 \end{pmatrix}$$ w.r.t. the standard basis

Rmk: for the next week, it is possible to show that $J_+(\mathbb{R}^4)$ is a submanifold of $O(n)$ via

$$f : O(n) \to O(n), \quad A \to A^2$$

show that $-Id$ is a regular value of f + submersion thm.

2) $G(4,2) \simeq \mathcal{J}_0(\mathbb{R}^4) \times \mathcal{J}_1(\mathbb{R}^4)$ (Sketch)

Pf: Let $L = \operatorname{span}(e_1, e_2)$ be an oriented 2-dim vspace. Then

$$\mathbb{R}^4 = L \oplus L^\perp$$

Choose a basis (onb) of $L^\perp$ s.t. e_1, e_2, f_1, f_2 has the same orientation of $\mathbb{R}^4$. Then let

$$\begin{aligned} \Pi_0 : G(4,2) &\longrightarrow \mathcal{J}_0(\mathbb{R}^4) \\ L &\longrightarrow \mathcal{J}_{L,0} \end{aligned}$$

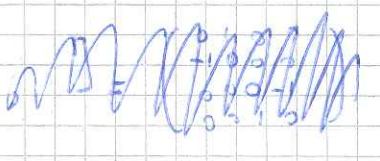
$$\mathcal{J}_{L,0}(e_1) = e_2 \quad , \quad \mathcal{J}_{L,0}(e_2) = -e_1$$

$$\mathcal{J}_{L,0}(f_1) = f_2 \quad , \quad \mathcal{J}_{L,0}(f_2) = -f_1$$

+ oriented

Π_0 is surj since $\forall$ any $\mathcal{J} \in \mathcal{J}_0(\mathbb{R}^4)$ there exists a basis (onb) s.t.

$$\mathcal{J} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$



Analogously define $\Pi_1 : G(4,2) \longrightarrow \mathcal{J}_1(\mathbb{R}^4)$

$$\begin{aligned} L &\longrightarrow \mathcal{J}_{L,1} \end{aligned}$$

$$\mathcal{J}_{L,1}(e_1) = -e_2 \quad , \quad \mathcal{J}_{L,1}(e_2) = e_1$$

$$\mathcal{J}_{L,1}(f_1) = -f_2 \quad , \quad \mathcal{J}_{L,1}(f_2) = f_1$$

Again Π_1 is surj since $\forall$ any $\mathcal{J} \in \mathcal{J}_1(\mathbb{R}^4)$ there exists a basis s.t.

$$\mathcal{J} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that Π_0, Π_1 depends by the choice of the basis of $L^\perp$,

$$\Pi_0 \times \Pi_1 : G(4,2) \longrightarrow \mathcal{J}_0(\mathbb{R}^4) \times \mathcal{J}_1(\mathbb{R}^4)$$

is a big.

Repetition of Topology

In the first part we recall some facts about topology.
All these facts may be found in Munkres's Topology. We start with some definitions.

Def: Let X be a topological space, a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to $x\in X$ (we denote $x_n \to x$) if:

$$\forall U \text{ open}, x\in U \quad \exists N\in\mathbb{N} \text{ s.t. } x_m\in U \quad \forall m\geq N$$

Rmk: The limit as defined is in general not unique. But if X is an Hausdorff space, then the limit is unique

Ex: • Consider $\mathbb{R}$ equipped with the trivial topology $\mathcal{T}=\{\emptyset,\mathbb{R}\}$ Then any point of $\mathbb{R}$ is the limit of the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$.

• $\mathbb{R}$ equipped with its standard topology (hence Hausdorff), then 0 is the unique limit of $\{\frac{1}{n}\}_{n\in\mathbb{N}}$.

Another important definition is the following

Def: Let X be a topological space. A neighborhood basis of X is a family of open sets $\mathcal{B}_p$ defined for any $p\in X$, such that

$$(*)\quad \forall p\in X, \forall U \text{ open}, p\in U \Rightarrow \exists B\in\mathcal{B}_p, B\subset U$$

X is said to be first countable if each $\mathcal{B}_p$ is a countable set.

Thm: (Thm 30, Munkres, p. 190)
Let X, Y be a topological spaces

① $A\subset X$, if there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ in A converging to x, then $x\in\bar{A}$
If X is first countable then the reverse is true.

② $f: X\to Y$. If f is continuous, then for every sequence $x_i\to x$ in X, then $f(x_i)\to x$.
If X is first countable then the reverse is true.

pf: ① **Claim 1**: $A\subseteq X$, $x\in\bar{A} \iff \forall U$ open, $x\in U$ we have $U\cap A\neq\emptyset$.

pf: we show:

$$x\notin\bar{A} \iff \text{there exist } U \text{ open}, x\in U \text{ s.t. } U\cap A=\emptyset$$

"$\Rightarrow$" $x\notin\bar{A}$, set $U:=X/\bar{A}$ open, then $(X/\bar{A})\cap\bar{A}=\emptyset$.

"$\Leftarrow$" Recall $\bar{A}:=\bigcap_{\substack{C \text{ closed}\\ C\supset A}} C$. Since X/U is closed and $X/U\supset A$, $x\notin X/U \Rightarrow x\notin A$.

□ Claim 1

Let X be a top space $A \subset X$. Let $\{x_i\}_{i\in\mathbb{N}}$ in A $x_i \to x \in X$.
Then $\forall\, U$ open, $x \in U$, then exist. some x_i in U and in particular
$U \cap A \neq \emptyset$. Hence by claim 1, $x \in \bar{A}$.
Let X be a first countable topological space, let $x \in \bar{A}$, $A \subset X$. By Claim 1
we have $B \cap A \neq \emptyset$ for any $B \in \mathcal{B}_x$. Choose a $B_0 \in \mathcal{B}_x$ and choose an
$x_0 \in B_0 \cap A$.
Since B_0 is open, by (*) there exists a $B_1 \in \mathcal{B}_x$, $B_1 \subset B_0$. Choose an
$x_1 \in B_1 \cap A$, and so on ...
It is then very simple show that $x_i \to x$.

(1)

we prove (2). Let $f: X \to Y$ be continous, $x_i \to x \in X$. Consider $\{f(x_i)\}_{i\in\mathbb{N}}$ in Y.
Let V be an open set in Y, $f(x) \in V$. Then $f^{-1}(V)$ is an open
set containing x. Then there exists a $N \in \mathbb{N}$ s.t $x_m \in f^{-1}(V)$, for $m \geq N$.
Then $f(x_m) \in V$ for $m \geq N \Rightarrow f(x_i) \to f(y)$ in Y.

Let X be a first countable topological space. Let $f: X \to Y$ s.t if $x_i \to x$ in X then
$f(x_i) \to f(x)$ in Y. Recall that a function is continuous iff: $\forall A \subset X$, $f(\bar{A}) \subseteq \overline{f(A)}$
(Equivalent def of continuity, see Munkres on Wikipedia ☺)
Then let $x \in \bar{A}$, thus $\exists\, x_i \to x$ in X (1). Thus: $f(x_i) \to f(x) \overset{(1)}{=_D} f(x) \in \overline{f(A)}$

(2)

Another technical tool is the following

Lemma: Let X be a topological space. Let $K \subset X$, K compact. Then any sequence
$\{x_i\}_{i\in\mathbb{N}}$ in K has a convergent subsequence

Pf: Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence with no convergent subsequence. Then there exi
Let $x \in K$, U open in K U contains finitely many x_i. (otherwise we can build
a subsequence $x_{i_j} \to x$.
Thus for any $x \in K$ we can build the open set U_x that contains only finitely many
x_i. Let $\mathcal{U} = \{U_x \mid x \in K\}$, clearly it is an open cover of K but there exists
no finite subcover.

Now we stop our topology part and we will start the exercise 2.a)

3a)

Claim 1 : Any smooth manifold M is a first countable space

pf: Let $p \in M$, choose a chart $\psi : U \to \mathbb{R}^n$, $p \in U$, $\tilde{p} := \psi(p)$
Let $B_r(\tilde{p})$ be the open ball of radius r in $\mathbb{R}^n$. Then there exist a $\delta > 0$ s.t $B_\delta(\tilde{p}) \subset \psi(U)$. Define $\mathcal{B}_p$ as

$$\mathcal{B}_p := \left\{ \psi^{-1}\left(B_{\delta \cdot t}(\tilde{p})\right) \;\middle|\; t \in (0,1] \cap \mathbb{Q} \right\}$$

it is clearly countable and a nbhd basis

∎ Claim 1

Claim 2: Let $f : M \to N$ be proper smooth map between smoth mfld, then it is a closed map.

pf: Let C be a closed subset of M. Let $y \in \overline{f(C)}$. By the theorem above there exists a sequence $\{y_i\}_{i \in \mathbb{N}}$ s.t $y_i \to y$.
Let $x_i \in C$ s.t $f(x_i) = y_i$.
Let $\psi : U \to \mathbb{R}^n$ be a charts of N, $p \in U$. Then there exist an $r > 0$ s.t $B_r(\tilde{p}) \subset \psi(U)$, $\tilde{p} := \psi(p)$.
Let $V := \psi^{-1}\left(B_{\frac{r}{2}}(\tilde{p})\right)$. Since $\overline{V} = \psi^{-1}\left(\overline{B_{\frac{r}{2}}(\tilde{p})}\right)$ and $\overline{B_{\frac{r}{2}}(\tilde{p})}$ is compact in $\mathbb{R}^n$ then $\overline{V}$ is compact.
For large enough i, $y_i \in U \subset \overline{U}$, and therefore $x_i \in F^{-1}(\overline{U})$, $i \geq N$
Since f is proper we have $f^{-1}(\overline{U})$ is compact and thus $\{x_i\}_{i \geq N}$ as a convergent subsequence $\{x_{i_k}\}$ that converge to some $\tilde{x} \in M$.
C is closed $x_{i_k} \in C$, then $\tilde{x} \in C$. Since f is continuous:

$$f(\tilde{x}) = \lim_{k \to \infty} f(x_{i_k}) = \lim_{k \to \infty} (y_{i_k}) = y$$

then since $f(\tilde{x}) \in f(C)$ we have $y \in f(C)$

∎ Claim 2

Claim 3: Let $f : M \to N$ be a inj proper immersion, then f is an embedding.

pf: By the lecture we know:
(*) if $g : M \to N$ inj immersion, g homeom onto its image $\Rightarrow g(M)$ submfld of N, g diffeom.

We show that f is an homeomorphism onto its image. Clearly $f : M \to f(M)$ is inj and surj and continuous. We must show that $f^{-1} : f(M) \to M$ is continuous.

By claim 2 f is a closed map, thus:

Let $U \subset M$ be an open set in M, then $f(M \setminus U)$ is closed in $f(M)$. Since f is bijective $f(M \setminus U) = f(M) \setminus f(U)$. Thus

$$f(U) = \left(f(M \setminus U)\right)^c = \left(f(M) \setminus f(U)\right)^c \quad \text{is open in } f(M).$$

Thus f is an open map $\Rightarrow f^{-1}$ is continuous ∎ Ex. 2a

(4) Orientierung:

For a nice discussions about orientation consider
Lee, Intro to C^∞ mnfld, page 327.

Pf: Recall that two bases $(e_1,\dots,e_n)$ and $(e_1',\dots,e_n')$ are similarly oriented if the change of coordinates matrix:

$$A(e_i) = e_j'$$

has $\det A > 0$

(otherwise we say that they are negatively oriented)

An orientation of V is an element of

$$\{\tau_1,\tau_2\} = \{\text{set of bases } V\}/\sim \qquad (e_1,\dots,e_n)\sim(e_1',\dots,e_n') \iff \text{similarly oriented.}$$

An orientation of M is a family $\{\tau_q\}_{q\in M}$ s.t.

- τ_q is an orientation of T_qM
- $\{\tau_q\}_{q\in M}$ is locally constant, i.e.: $\forall p\in M$ $\exists U$ open $p\in U$ s.t.
  $\exists$ a frame, i.e. a smooth map $(e_1,\dots,e_n): U \to TM|_U$

  s.t. each $(e_1(q),\dots,e_n(q))$ is an ~~orientation~~ basis of T_qM with orientation τ_q.

Here: Let M be a mnfld, for any p choose a chart $\psi: U \to \mathbb{R}^n$. Then
Consider the standard global frame on $\mathbb{R}^n$. i.e

$$e_1 : (x_1,\dots,x_n) = \left((x_1,\dots,x_n), \frac{\partial}{\partial x_1}\right) \in T\mathbb{R}^n$$

$$\vdots$$

$$e_1(x_1,\dots,x_n) = \left((x_1,\dots,x_n), \frac{\partial}{\partial x_n}\right)$$

Since $\psi: U \to \psi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism we have that ψ^{-1} induces a

$$\tilde e_1^\psi,\dots,\tilde e_n^\psi : U \longrightarrow TM \quad \text{via}$$

$$\tilde e_i(q) := (d\psi_q)^{-1}\left(\frac{\partial}{\partial x_i}\right)$$

Now let τ_q^ψ be the orientation of $\left(\tilde e_1^\psi(q),\dots,\tilde e_n^\psi(q)\right)$, $q\in U$

Observation

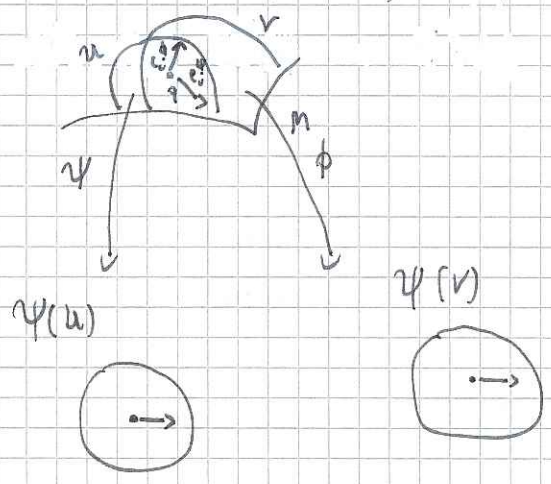
① This assignment is **not** a local frame, in particular because $\tau_q^{\psi} \neq \tau_q^{\phi}$ where $\phi: V \to \mathbb{R}^n$ is another chart, $q \in V \cap U$.

In particular we have:

$$\tilde{e}_j^{\psi}(q) = \left(d\psi_q^{-1}\right)\left(\frac{\partial}{\partial x^j}\right)$$

$$\tilde{e}_j^{\phi}(q) = \left(d\phi_q^{-1}\right)\left(\frac{\partial}{\partial x^j}\right)$$

Thus:



it follows: $\tilde{e}_j^{\psi}(q) = d\psi_q^{-1}\, d\phi_q\, \tilde{e}_i^{\phi}$

② Thus if $\det\left(D\left(\phi \circ \psi^{-1}\right)\right) > 0$ wherever for all the charts of M then the above construction is a local frame and it defines an orientation of M.

Now consider $TM = \bigcup_{p \in M} T_pM$. For a chart $\psi: U \to \mathbb{R}^n$ we define

$$\mathcal{U} := TU = \bigcup_{p \in U} T_pM \quad \text{and}$$

$$\Psi: \underset{(p,X)}{TU} \longrightarrow \underset{(x^1,\dots,x^n,X^1,\dots,X^n)}{\psi(U) \times \mathbb{R}^n}, \qquad X = \sum X^i \left(\frac{\partial}{\partial x^i}\right)_{\psi,p}$$

Recall that Ψ is obtained from:

$$\psi: \underset{p}{U} \longrightarrow \underset{(x^1,\dots,x^n)}{\psi(U)}, \qquad d\psi_p: T_pM \longrightarrow (X^1,\dots,X^n), \quad \sum X^i \left(\frac{\partial}{\partial x^i}\right)_{p,\psi}$$

Thus it follows that $d\Psi_{(p,X)} = \begin{pmatrix} d\psi_p & 0 \\ 0 & d\psi_p \end{pmatrix}$

Now let $\tilde{e}_j^{\Psi}$ as above, $j = 1,\dots,2n$, then for a chart $\phi: V \to \mathbb{R}^n$ consider the chart $\Phi: \mathcal{V} \to \phi(V) \times \mathbb{R}^n$

Thus: $\tilde{e}_j^{\,\Psi}(p,x) = \underbrace{\left(d\,\Psi_{(p,x)}\right)^{-1} d\,\Phi_{(p,x)}}_{A}\ \tilde{e}_j^{\,\Phi}$

Since $A = \begin{pmatrix} d\psi_1^{-1} \circ d\,\Phi_1 & 0 \\ 0 & d\psi_1^{-1} \cdot d\,\Phi_1 \end{pmatrix} \Rightarrow \underline{\det(A) > 0}$

$\Rightarrow \tilde{e}_j^{\,\Psi}$ $j = 1, \dots, 2n$ is a local frame

Rmk: Observation (1) follows:

if M is a smooth mfld s.t $\det\left(d(\phi \cdot \psi^{-1})\right) > 0$ for all the charts

$\Rightarrow M$ is orientable

The reverse is also true:

M orientable $\Rightarrow$ there exists an atlas $\mathcal{A}$ on M s.t

- $\phi, \psi \in \mathcal{A} \Rightarrow \det\left(d(\phi \cdot \psi^{-1})\right) > 0$
- $\mathcal{A}$ is compatible with the atlas of M.

See Lee, Intro to C^∞ mfld, Prop 13.3.

5) Quaternions

a) $S^3 = \{p \in \mathbb{Q} \mid |p| = 1\}$

1) $Ad : S^3 \longrightarrow Aut(\mathbb{R}^3)$ is a group action

pf: $\mathbb{R}^3 \cong \{ai + bj + ck \mid a,b,c \in \mathbb{R}\}$ {Δ sometimes the people use R instead $\mathbb{R}^3$}

Claim: $\mathbb{R}^3 = \{p \in \mathbb{Q} \mid \bar{p} = -p\}$

pf: $\subseteq$ is clear. Let $p,q \in \{p \in \mathbb{Q} \mid \bar{p} = -p\}$, then easily we have:

$\mu q + \lambda p \in \{p \in \mathbb{Q} \mid \bar{p} = -p\}$ $\quad \lambda, \mu \in \mathbb{R}$.

Since $\mathbb{R}i, \mathbb{R}j, \mathbb{R}k \subseteq \{p \in \mathbb{Q} \mid \bar{p} = -p\}$ it follows

$\mathbb{R}^3 = \{p \in \mathbb{Q} \mid \bar{p} = -p\}$.

□ Claim.

Let $u \in S^3$, then for $p \in \mathbb{R}^3$

$$\overline{Ad_u(p)} = \overline{upu^{-1}} = \bar{u}^{-1}\bar{p}\bar{u} = \frac{\bar{u}}{u\cdot\bar{u}}(-p)\bar{u} = -u \cdot p \cdot u^{-1}$$

$$= -(upu^{-1}).$$

$\Rightarrow Ad_u$ is well def.

- Let $1 \in S^3$, $Ad_1(p) = p$ ✓
- Let $v_1, v_2 \in S^3$, $Ad_{v_1} Ad_{v_2}(p) = v_1 v_2 p v_2^{-1} v_1^{-1} = v_1 v_2 p (v_1 v_2)^{-1} = Ad_{v_1 v_2}(p)$

□ 1)

2) Ad_v is an isometry $\forall v \in S^3$

$$|Ad_v(p)| = |v|\,|p|\,|\bar{v}^{-1}| = |p|$$

□ a)

b) We denote with $so(3)$ the so called Lie algebra of $SO(3)$ you will see in the next exercise sheet what it is. for the moment

$$so(3) = \{A \in M^{3\times 3} \mid A^T = -A\}$$

Consider the map $\mathbb{R}^3 \longrightarrow so(3)$

$$u = ia + jb + kc \longmapsto \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} =: \hat{u}$$

Note that:

$$(\hat{v})^2 = \begin{pmatrix} -b^2-c^2 & ab & ac \\ ab & -a^2-c^2 & bc \\ ac & bc & -a^2-b^2 \end{pmatrix}$$

$$(\hat{v})^3 = -(a^2+b^2+c^2)\,\hat{v} \qquad \text{thus:}$$

Let $e^{t\hat{v}}$ be the matrix exponential, since $\det\left(e^{t\hat{u}}\right) = e^{t\,\mathrm{tr}(t\hat{u})} = e^{(0)} = 1$

$$\left(e^{t\hat{v}}\right)\cdot\left(e^{t\hat{v}}\right)^T = e^{t\hat{v}}\cdot e^{(t\hat{v})^T} = e^{t\hat{v}}\cdot e^{-t\hat{v}} = e^{t\hat{v}-t\hat{v}} = e^0 = \underline{1} \Rightarrow e^{t\hat{v}} \in SO(3)$$

Then for $v \in \mathbb{R}^3 \cap S^3$ we have:

$$e^{t\hat{v}} = \sum_{k\geq 0} \frac{t^k \hat{v}^k}{k!} = \underline{1} + \sum_{k\geq 0} \frac{t^{2k+1}\hat{v}^{2k+1}}{(2k+1)!} + \sum_{k\geq 1} \frac{t^{2k}\hat{v}^{2k}}{(2k!)}$$

$$= \underline{1} + \sum \frac{t^{2k+1}(\hat{v})\cdot(-1)^k}{(2k+1)!} + \sum_{k\geq 1} \frac{t^{2k}\hat{v}^2(-1)^k}{(2k)!}$$

$$= \underline{1} + \sin(t)\,\hat{v} + \left(\cos(t)-1\right)\hat{v}^2$$

$$= \underline{1} + \sin(t)\hat{v} + -2\sin^2(t/2)\,\hat{v}^2$$

Now let $u \in Q$ $|u|=1$. Then we can write

$$u = \cos(t/2) + \sin(t/2)v \qquad , v \in \mathbb{R}^3 \qquad , |v| = 1.$$

Now: $Ad_u(w) = uwu^{-1} = \left(\cos(t/2) + \sin(t/2)v\right) w \left(\cos(t/2) - \sin(t/2)v\right)$

$$= \cos^2(t/2)\,w + \sin(t/2)\cos(t/2)\,(vw - wv) - \sin^2(t/2)\,vwv$$

An explicit calculation show that:

$$vw - wv = 2\hat{v}w \qquad , \qquad vwv = -2\hat{v}^2 w - w$$

In particular: $Ad_u(w) = \cos(t/2)\,w + \sin(t/2)\cos(t/2)\,2\hat{v}w + \sin^2(t/2)\left[2\hat{v}w + w\right]$

$$= w + \sin(t)\,\hat{v}w + 2\sin^2(t/2)\,\hat{v}^2 w$$

$$= \underline{e^{t\hat{v}}\,w} \qquad \forall w \in \mathbb{R}^3$$

By lin Alg we know that $e^{t\hat{v}}$ represent a rotation along the axis v of angle t.

1) By show $e^{t\hat{v}} \in SO(3)$ $\Rightarrow$ it is orthogonal map. Since $u = \cos\frac{t}{2} + \sin\frac{t}{2}v$, $v \in \mathbb{R}^3$, $|v|=1$, $u \in S^3$ and $Ad_u(w) = e^{t\hat{v}}(w)$, for any $w \in \mathbb{R}^3$, then we have that

$Ad_u(v) = v$, $Ad_u(-v) = -v$ $\Rightarrow$ we have the axis!

Note for $u = \sin(\frac{t}{2}) + \cos(\frac{t}{2})v$ we have again

$Ad_{u(t)}(v) = v$, $Ad_{u(t)}(v) = -v$

$\Rightarrow \forall t \in [0, 2\pi]$, $e^{t\hat{v}}$ is a rotation along v in $\mathbb{R}^3$

2) For the angle?

By an explicit calculation we consider that $e^{t\hat{v}}(-)$ is a rotation along v with angle t.

3) For $u = 1 + 0\cdot i + 0\cdot j + 0\cdot k \Rightarrow v = (0,0,0)$ $e^{2\pi \cdot \hat{0}} = \text{identity}$

c) $Ad : S^3 \longrightarrow SO(3)$

$v \longrightarrow e^{t\hat{u}}$

Claim: Ad is a group homomorphism of groups, and it surj.

pf: • The surjectivity is clear. Let $A \in SO(3)$, then $A = R_t(u)$ $u \in \mathbb{R}^3$ $|u| = 1$

Let $V_1 := \cos(t/2) + \sin(t/2)\cdot u_2$, then

$V_2 := \cos(t'/2) + \sin(t'/2)\cdot u_2$

Then $Ad_{(V_1 \cdot V_2)} = (Ad_{V_1})(Ad_{V_2})$?

$\Rightarrow$ This follows from part a)

Invariance

Claim

Claim: $Ad : S^3 \longrightarrow SO(3)$ is a two sheeted covering map

pf: Consider the map: $h : S^3 \longrightarrow \mathbb{R}^3$

$$h(a+bi+cj+dk) = \begin{cases} (bi+cj+dk)/\sqrt{1-a^2} & a \neq 1 \\ (0,0,0) & a = 1 \end{cases}$$

Let $Ad : S^3 \longrightarrow SO(3) \overset{\wedge}{h(v)}$

$v \longrightarrow e$

is the composition of:

$S^3 \longrightarrow \mathbb{R}^{3\times 3} = \mathbb{R}^6$ $v \longrightarrow (h(v))$ smooth

$\mathbb{R}^6 = \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}^{3\times 3} = \mathbb{R}^6$ $A \longrightarrow \exp(A)$ smooth

$\Rightarrow Ad$ is smooth

Now: $T_{id}\,SO(3) = T_{id}\,O(3)$, this because:

$\det : O(3) \longrightarrow \mathbb{R}$ is continuous

and $\det^{-1}(\underbrace{\{x \in \mathbb{R} \mid x > 0\}}_{\text{open}}) = SO(3)$ is open.

By ex sheet 3, 1.3: $B \in T_{id}\,O(3) \iff B^T + B = 0$ and $\gamma(t) := e^{tB}$ is a smooth path s.t: $\gamma(0) = Id$, $\dot\gamma(0) = B$.

It follow easily that:

- $v \in S^3 \cap \mathbb{R}^3 \Rightarrow \hat v^T + \hat v = 0$.
- $T_{id}\,SO(3)$ is generated by $A_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Consider the three paths:

$\mu_1 : (-\varepsilon, \varepsilon) \longrightarrow S^3$, $\mu_1(t) = \cos(\frac{t}{2}) + \sin(\frac{t}{2}) \cdot i$, $\dot\mu_1(0) = i/2$

$\mu_2 : (-\varepsilon, \varepsilon) \longrightarrow S^3$, $\mu_2(t) = \cos(\frac{t}{2}) + \sin(\frac{t}{2}) \cdot k$, $\dot\mu_2(0) = k/2$

$\mu_3 : (-\varepsilon, \varepsilon) \longrightarrow S^3$, $\mu_3(t) = \cos(\frac{t}{2}) + \sin(\frac{t}{2}) \cdot i$, $\dot\mu_3(0) = i/2$

it follows:

$Ad_{\mu_1(t)} = e^{t \cdot A_1}$, $Ad_{\mu_2(t)} = e^{t \cdot A_2}$, $Ad_{\mu_3(t)} = e^{t A_3}$

We conclude that $d\,Ad_{Id} : T_1 S^3 \longrightarrow T_{id}\,SO(3)$ is surjective and linear

Since: $T_1 S^3 \overset{3.2}{\cong} \mathbb{R}^3$, $T_{id}\,SO(3) = T_{id}\,O(3) \cong \mathbb{R}^{\frac{3\cdot 2}{2}}$

$\Rightarrow$ Ad_{Id} is an isomorphism.

Since: $SO(3)$, S^3 are lie groups and Ad is a lie group homomorphism we conclude

- Let $L_v : S^3 \longrightarrow S^3$, $w \longrightarrow vw$ be the left multiplication, clearly it is a smooth diffeomorphism.
- $d\,Ad_v = d L^{-1}_{Ad_v} \circ d\,Ad_{Id} \circ d L_v$ i.e:

$$\begin{array}{ccc} T_{(1,0,0,0)} S^3 & \xrightarrow{d\,Ad_{Id}} & T_{id} SO(3) \\ \uparrow d L_v \ (\text{from } (k)) & (2) & \uparrow d L_{(Ad_v)} \ (\text{from } (k)) \\ T_v S^3 & \longrightarrow & T_{Ad_v} SO(3) \end{array}$$

$\Rightarrow$ $d\,Ad_v$ is an isomorphism.

Now: $Ad : S^3 \longrightarrow SO(3)$ is a local diffeom.

Since:

- $\dim(Q) = \dim SO(3)$
- S^3 is compact in $\mathbb{R}^4$

Ex Sheet 6, Ex 1

$\Rightarrow$ Ad is a covering map, where:

$$(Ad)^{-1}(A) = \{v, -v\} \quad , A = e^{t\hat{u}} \quad , v = \cos(t/2) + \sin(t/2)\, u \quad |u| = 1.$$

Now: • Since $\pi : S^3 \longrightarrow \mathbb{R}P^3$, $v \longrightarrow [v]$ is smooth 2 sheeted covering map, we define

$\phi : SO(3) \longrightarrow \mathbb{R}P^3$, $\phi(A) = \pi(Ad^{-1}(A))$

Note that

ϕ is well def: $\phi(A) = \pi(Ad^{-1}(A)) = \pi(\{v, -v\}) = [v]$

ϕ^{-1}, ϕ is smooth: Ad, π are local diffeom Ad^{-1}, π^{-1} locally exists!

$\Rightarrow \phi$ is a diffeom.

$\blacksquare$ 2c)

a) Let $US^2 := \coprod_{x \in S^2} \{v \in T_x S^2 \mid |v| = 1\}$

Note that:

Since $(x, v) \in US^2$, then we get a $w = \hat{x}(v) = x \times v$ [note that we consider $x \in S^2$ as a vector in $\mathbb{R}^3$] clearly $(x, v, x \times v)$ is an orthonormal basis of $\mathbb{R}^3$, thus:

$$US^2 \longrightarrow \text{ONB of } \mathbb{R}^3$$

Analogously, given a ONB (v_1, v_2, v_3) of $\mathbb{R}^3$ then it defines a point (v_1, v_2) of US^2, where v_1 is considered as a point of S^2, thus

$$\text{ONB of } \mathbb{R}^3 \longrightarrow US^2$$

Thus: ONB of $\mathbb{R}^3 \simeq US^2$. Let (e_1, e_2, e_3) be the standard basis of $\mathbb{R}^3$.

Consider $\Phi : SO(3) \longrightarrow US^2$

$$A \longrightarrow (Ae_1, Ae_2)$$

$\Phi^{-1} : US^2 \longrightarrow SO(3)$

$(x, v) \longrightarrow A_{(x,v)}$ def by $A_{(x,v)}(e_1) = x$, $A_{(x,v)}(e_2) = v$, $A_{(x,v)}(e_3) = x \times v$

Note that it is easy show that Φ is smooth, in particular

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

[it is really formal!]

Conclusion: $US^2 \simeq SO(3)$

IV d)

e) Recall that the Hopf fibration:

$$S^3 \subset \mathbb{C}^2 \longrightarrow S^2 \subset \mathbb{R}\times\mathbb{C}$$
$$(z_1, z_2) \longrightarrow \left(|z_1|^2 - |z_2|^2,\ 2 z_1 \bar{z}_2\right)$$

Now: $S^3 \xrightarrow{Ad_v} SO(3) \xrightarrow{\Phi} US^2 \xrightarrow{\pi} S^2$

where: $\pi: US^2 \longrightarrow S^2$, $(x, v) \longrightarrow x$

Then: $v \longrightarrow Ad_v \longrightarrow (Ad_v e_1, Ad_v e_2) \longrightarrow Ad_v e_1$

Using the Quaternion we have:

- $v \in Q$, $v = a + ib + jc + kd$, $Q \overset{\psi_1}{\simeq} \mathbb{C}\times\mathbb{C}$ $\quad (a+ib+jc+kd) \simeq (a+ib,\ c+id)$
- $\mathbb{R}^3 \ni v \in \mathbb{R}^3$, $v = ai + bj + ck$, $\mathbb{R}^3 \overset{\psi_2}{\simeq} \mathbb{R}\times\mathbb{C}$ $\quad (ai+jb+ck) \simeq (a,\ b+ik)$

Then we have:

- If we consider $\mathbb{R}^3$ with quaternion, then the standard basis is given by (i, j, k)

Conclusion:

$$\begin{aligned} \pi \circ \phi \circ Ad(v) &= Ad_v(e_1) \qquad e_1 = i \\ &= (a+ib+jc+kd)\, i\, (a-ib-jc-kd) \\ &= i(a^2+b^2-c^2-d^2) + j(2(cd+bc)) + k(2(bd-ac)) \end{aligned}$$

Then:

$$\left\{(z_1, z_2) \in \mathbb{C}^2 \,\middle|\, |z_1|^2 + |z_2|^2 = 1\right\} \overset{\psi_1}{\simeq} S^3 \xrightarrow{\pi\circ\phi\circ Ad_v} S^2 \overset{\psi_2}{\simeq} \left\{(h, t) \in \mathbb{R}\times\mathbb{C} \,\middle|\, h^2 + |t|^2 = 1\right\}$$

and $$\psi_2 \circ (\pi \circ \phi \circ Ad_v)\, \psi_1 (z_1, z_2) = \left(|z_1|^2 - |z_2|^2,\ 2 z_1 \bar{z}_2\right)$$