Exercise Sheet 6

Exercise 1

(a) Let G be a connected topological group, and let H be a discrete normal subgroup of G. Show that H is contained in the center $Z(G)$ of G.

(b) Use (a) to prove that the fundamental group of a Lie group is abelian.

Exercise 2

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let H be a closed normal subgroup of G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Show that G/H, with the smooth structure given by Theorem 2.14 and with its natural group structure, is a Lie group with Lie algebra $\mathfrak{g}/\mathfrak{h}$.

Exercise 3

Find a simple proof of the fact that every finite-dimensional Lie algebra with trivial center is the Lie algebra of some Lie group.

Exercise 4

Let G be a Lie group with Lie algebra $\mathfrak{g} = TG_e$.

(a) Show that if $X, Y \in \mathfrak{g}$ and $[X, Y] = 0$, then $exp(X + Y) = exp(X)exp(Y)$.

(You may use the fact, shown in class, that a Lie group is abelian if and only if its Lie algebra is abelian.)

(b) Conversely, show that if exp is a homomorphism from $(\mathfrak{g}, +)$ to G, then $\mathfrak{g}$ is abelian.

(Hint: Look at the differential of exp at 0.)