Exercise Sheet 7

Exercise 1

Show that every n-dimensional connected abelian Lie group is isomorphic to $(S^1)^r \times \mathbb{R}^{n-r}$ for some $0 \leq r \leq n$.

Exercise 2

Prove the following basic properties of ideals of a Lie algebra $\mathfrak{g}$:

(a) If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathfrak{g}$, so are $\mathfrak{a} + \mathfrak{b}$ and $[\mathfrak{a}, \mathfrak{b}]$ (the subspace generated by all $[A, B]$ with $A \in \mathfrak{a}$ and $B \in \mathfrak{b}$).

(b) If $\phi\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then $ker(\phi)$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}/\ker(\phi)$ is isomorphic to $\phi(\mathfrak{g})$.

(c) If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathfrak{g}$ with $\mathfrak{a} \subset \mathfrak{b}$, then $\mathfrak{g}/\mathfrak{b}$ is isomorphic to $(\mathfrak{g}/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a})$.

(d) If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{b}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{a} \cap \mathfrak{b}$ is an ideal of $\mathfrak{a}$, $\mathfrak{b}$ is an ideal of $\mathfrak{a} + \mathfrak{b}$, and $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$ is isomorphic to $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$.

Exercise 3

Consider the Heisenberg group

$$G := \left\{ \left(\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \right) : x, y, z \in \mathbb{R} \right\} \subseteq \mathrm{GL}(3, \mathbb{R})$$

and the subgroup

$$H := \left\{ \left(\begin{pmatrix} 1 & 0 & m \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right) : m \in \mathbb{Z} \right\}$$

of G. Check that G/H is a connected, solvable Lie group and show that G/H does not admit a smooth, injective homomorphism into $\mathrm{GL}(V)$ for any finite-dimensional $\mathbb{C}$-vector space V.

Exercise 4

Prove that the Lie group $Aff(1)$ of the affine transformations of the real line is solvable but not nilpotent.