# Solutions Exercise Sheet 2

## Exercise 1

Let $G$ be a locally compact topological group, and let $\mu$ be a left Haar measure on $G$.

(a) $G$ is discrete if and only if $\mu(\{e\}) > 0$.

*SOLUTION*:

If $G$ is discrete then $\{e\}$ is open and since $\mu$ is a Haar measure, we get that $\mu(\{e\}) > 0$.

Now assume $\mu(\{e\}) > 0$ and let $K$ be a compact neighborhood of $e$. Let $U$ be the open interior of $K$. Since $K$ is compact and $\mu$ is Haar then $\mu(K) < \infty$ and hence $\mu(U) < \infty$.

Set $m = \mu(\{e\}) > 0$. Because all translation functions are homeomorphism, we get that $\mu(\{x\}) = m > 0$ for all $x \in G$. Combining this with the fact that $\mu(U) < \infty$ we immediately obtain that $U$ is finite. Since $e \in U$ we get that $U = \{e, x_1, \ldots, x_n\}$ for some $x_i \in G, 1 \leq i \leq n$. But then $\{e\} = U \cap \{x_1, \ldots, x_n\}^c$. Hence $\{e\}$ is open and by virtue of the translation homeomorphisms we get that $G$ is discrete.

(b) $G$ is compact if and only if $\mu(G) < \infty$.

*SOLUTION*:

If $G$ is compact then because $\mu$ is Haar we get that $\mu(G) < \infty$.

Now assume $\mu(G) < \infty$. Let $K$ be a compact neighborhood of $e$. The idea is to try to cover $G$ with finitely many disjoint translates of $K$. For this purpose set $K' = KK^{-1}$. Notice that $K'$ is compact being the image of the compact set $K \times K$ (check that this is compact, or ask me via email or in class why) through the continuous map $G \times G \to G, (x, y) \mapsto xy^{-1}$.

Notice that $K$ contains a non-empty open set so $\mu(K) > 0$. Since $K \subset K'$ we immediately get $\mu(K') > 0$. If $G$ is not compact then there exist a sequence of points $x_i \in G, i \geq 0$ with $x_0 = e$ such that $x_k \notin \bigcup_{i<k} x_i K'$. Otherwise $G$ would be a finite union of compact sets and hence compact.

Let's look at the translates $x_i K$. If $x_i K \cap x_j K \neq \emptyset$ for $i < j$ then $x_i k_1 = x_j k_2$ for some $k_1, k_2 \in K$. But then $x_j = x_i k_1 k_2^{-1}$ and so $x_j \in x_i KK^{-1}$ which contradicts our choice for $x_j$. Thus all these translates are disjoint. But then

$$\mu(G) \geq \sum_{i=0}^{i=N-1} \mu(x_i K) = N\mu(K)$$

Letting $N \to \infty$ gives us $\mu(G) = \infty$ which is a contradiction. Thus $G$ is compact.

## Exercise 2

Find an example of a locally compact topological group that contains a Borel set with finite left Haar measure but infinite right Haar measure.

*SOLUTION*:

Take

$$B = \left\{ \begin{pmatrix} x & y \\ 0 & \frac{1}{x} \end{pmatrix} : x \in \mathbb{R}_{>0}, y \in \mathbb{R} \right\}$$

Consider the following functionals on $C_c(B)$

$$\Lambda_{\text{left}}(f) = \int_{\mathbb{R}_{>0}} \int_{\mathbb{R}} f(X) \frac{d\lambda(y) d\lambda(x)}{x^2}$$

$$\Lambda_{\text{right}}(f) = \int_{\mathbb{R}_{>0}} \int_{\mathbb{R}} f(X) d\lambda(y) d\lambda(x)$$

where $d\lambda(y)d\lambda(x)$ is the standard Lebesgue measure on $\mathbb{R}^2$. Let's show $\Lambda_left$ is left invariant. We have that

$$\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \cdot \begin{pmatrix} x & y \\ 0 & \frac{1}{x} \end{pmatrix} = \begin{pmatrix} ax & ay + \frac{b}{x} \\ 0 & \frac{1}{ax} \end{pmatrix}$$

So the change of variables $\phi_1 : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x_{\text{new}}, y_{\text{new}})$ for the left translation is given by:

$$x_{\text{new}} = ax$$
$$y_{\text{new}} = ay + \frac{b}{x}$$

The Jacobian of this transformation is $J_1(x, y) = \begin{pmatrix} a & 0 \\ -\frac{b}{x^2} & a \end{pmatrix}$. Now let $L_{a,b}$ denote the left translation by the matrix $\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}$ We have that

$$\Lambda_{\text{left}}(f \circ L_{a,b}) = \int_{\mathbb{R}>0} \int_{\mathbb{R}} f(L_{a,b}X) \frac{d\lambda(y) d\lambda(x)}{x^2} =$$
$$= \int_{\mathbb{R}>0} \int_{\mathbb{R}} f(X \circ \phi_1(x, y)) \frac{d\lambda(y) d\lambda(x)}{x^2} =$$
$$= \int_{\mathbb{R}>0} \int_{\mathbb{R}} f(L_{a,b}X) a^2 \frac{d\lambda(y) d\lambda(x)}{(ax)^2} =$$
$$= \int_{\mathbb{R}>0} \int_{\mathbb{R}} f(L_{a,b}X) |\det(J_1)|^2 \frac{d\lambda(y) d\lambda(x)}{x_{\text{new}}^2} =$$
$$= \int_{\mathbb{R}>0} \int_{\mathbb{R}} f(X) \frac{d\lambda(y) d\lambda(x)}{x^2} = \Lambda_{\text{left}}(f)$$

This means that the functional $\Lambda_{\text{left}}$ defines a left Haar measure on $B$. Similarly, one can show that the functional $\Lambda_{\text{right}}$ defines a right Haar measure on $B$. Now consider the subset $B_1$ of $B$ defined by the inequalities $x \geq 1, 0 \leq y \leq 1$. It is easy to show that $B_1$ has finite left Haar measure but infinite right Haar measure.

## Exercise 3

Prove that the modular function $\Delta\colon G \to (0,\infty)$ (introduced in class) is a continuous homomorphism.

*SOLUTION*:

Let $\mu$ be a left Haar measure. Notice that

$$\Delta(gh) = \frac{\mu(Sgh)}{\mu(S)} = \frac{\mu(Sgh)}{\mu(Sg)}\frac{\mu(Sg)}{\mu(S)} = \Delta(h)\Delta(g) = \Delta(g)\Delta(h)$$

Hence $\Delta$ is a homomorphism. To show continuity, it is enough to show $\Delta$ is continuous at $e_G$. For this there are (at least) two ways of proceeding: either try to use Urysohn's lemma and the functional corresponding to $\mu$, or try to construct some open set $V$ and some measurable set $S$ for which $\mu(Sg)$ is "close" to $\mu(S)$ for all $g \in V$. We will take the later approach.

Pick $\varepsilon > 0$ and let $K$ be a compact neighbourhood of $e_G$ with non-empty open interior $U_0$ ($e_G \in U_0$). Set $K' = KK$. As in Exercise 1, $K'$ is compact. The idea is to now find as many measureble sets $S_i$ in between $K \subset K'$ and open sets $V_i$ with $e_G \in V_i$ such that $S_{i+1}V_{i+1} \subset S_i$. Then by a standard box principle type argument, the conclusion will follow.

Let $m : G \times G \to G$ be the multiplication map. Look at $m^{-1}(U_0)$. Since $e_G e_G = e_G$ we get that there exists open sets $V_0.V_0'$ with $e_G \in V_0, e_G \in V_0'$ such that $V_0V_0' \subset U$. Set $U_1 = V_0 \cap V_0' \cap U_0$. Since $e_G \in U_1$, $U_1$ is a nonempty open set with $U_1 \subset U_0$ and $U_1U_1 \subset U_0$. Switching $U_0$ with $U_1$, we can construct a nonempty open set $U_2$ with $e_G \in U_2$, $U_2 \subset U_1$ and $U_2U_2 \subset U_1$. Continuing inductively we get a sequence of nonempty open sets $(U_i)_i$ with $e_G \in U_i$, $U_{i+1} \subset U_i$ and $U_{i+1}U_{i+1} \subset U_i$. Moreover all $U_i$s are subset of $K$ which is compact so they have nonzero finite measure.

For $N > 0$, look at the sequence of inclusions

$$K \subset KU_N \subset KU_{N-1} \ldots \subset KU_1 \subset KU_0 \subset KK = K'$$

We have that

$$\mu(K') - \mu(K) \geq \mu(KU_0) - \mu(KU_N) = \sum_{i=0}^{N-1} \mu(KU_i) - \mu(KU_{i+1}) \geq 0$$

Pick $N > 0$ such that $0 \leq \dfrac{\mu(K') - \mu(K)}{N} < \varepsilon\mu(K)$. Since the $\mu(KU_i) - \mu(KU_{i+1})$ are $N$ non-negative real numbers adding up to at most $\mu(K') - \mu(K)$, it means that there exist some $i$ with $0 \leq i \leq N-1$ such that $0 \leq \mu(KU_i) - \mu(KU_{i+1}) \leq \dfrac{\mu(K') - \mu(K)}{N} < \varepsilon\mu(K)$. But then for all $g \in U_{i+1}$ we have that

$$\Delta(g) = \frac{\mu(KU_{i+1}g)}{\mu(KU_{i+1})} \leq \frac{\mu(KU_{i+1}U_{i+1})}{\mu(KU_{i+1})} \leq \frac{\mu(KU_i)}{\mu(KU_{i+1})} \leq \frac{\mu(KU_{i+1}) + \varepsilon\mu(K)}{\mu(KU_{i+1})}$$

Hence

$$\Delta(g) \leq 1 + \varepsilon\frac{\mu(K)}{\mu(KU_{i+1})} \leq 1 + \varepsilon\frac{\mu(K)}{\mu(K)} = 1 + \varepsilon$$

We should mention at this point that all sets $KU_i$ are measurable as they are open as union of open sets of type $kU_i$ with $k \in K$.

So $\Delta(g) \leq 1+\varepsilon$ for all $g \in U_{i+1}$. (1)

Consider the inversion map $inv : G \to G$ and set $U'_{i+1} = inv^{-1}(U_{i+1})$. We have that for all $g \in U'_{i+1}$

$$\Delta(g) = \frac{1}{\Delta(g^{-1})} \geq \frac{1}{1+\varepsilon} > 1-\varepsilon$$

The first inequality is due to the fact that for $g \in U'_{i+1}$ we have that $g^{-1} \in U_{i+1}$, and the second inequality is due to the fact that $1 > 1-\varepsilon^2 = (1-\varepsilon)(1+\varepsilon)$.

So for all $g \in U'_{i+1}$ we have that $\Delta(g) > 1-\varepsilon$. (2)

Combining (1) and (2) and setting $V = U_{i+1} \cap U'_{i+1}$ we get that for all $g$ in the neighborhood $V$ of $e_G$ the following inequality holds

$$1-\varepsilon < \Delta(g) \leq 1+\varepsilon$$

Continuity follows immediately.

## Exercise 4

Consider the locally compact Hausdorff group $G = (\mathbb{R}^n, +)$ where $n \in \mathbb{N}_0$.

(i) Show that $\mathrm{Aut}(G)$, i.e. the group of bijective homomorphisms which are homeomorphisms as well, is given by $\mathrm{GL}(n, \mathbb{R})$.

*SOLUTION*:

Pick $f \in \mathrm{Aut}(G)$. It is enough to show that $f$ is linear over the real vector space $\mathbb{R}^n$. Additivity is clear by virtue of being a morphism. We need to show $f(\lambda x) = \lambda f(x)$ for all $\lambda \in \mathbb{R}$. Because $f$ is a morphism, we immediately get $f(\lambda x) = \lambda f(x)$ for all $\lambda \in \mathbb{Z}$. Since

$$qf(\frac{p}{q}x) = f(px) = pf(x)$$

for $q \in \mathbb{Z}_{>0}$ and $p \in \mathbb{Z}$ we get that $f(\lambda x) = \lambda f(x)$ for all $\lambda \in \mathbb{Q}$.

Now pick $\lambda \in \mathbb{R}$ and consider an approximating sequence $(\lambda_i)_i$ of rational numbers. Then

$$f(\lambda x) = f(\lim_{i\to\infty} \lambda_i x) = \lim_{i\to\infty} f(\lambda_i x) = \lim_{i\to\infty} \lambda_i f(x) = \lambda f(x)$$

where the limit commutes with $f$ because of continuity. Thus $f$ is linear.

(ii) Show that $\mathrm{mod}_G : \mathrm{Aut}(G) \to \mathbb{R}_{>0}$ is given by $\alpha \mapsto |\det \alpha|^{-1}$.

*SOLUTION*:

Let $\mu$ be a left Haard measure on $G$ (guess what it is). Pick $A \in GL(n, \mathbb{R})$. Pick $f \in C_c(G)$. Let's evaluate

$$\int_{\mathbb{R}^n} f(Ax) d\mu(x)$$

Notice that the transformation $x \mapsto Ax$ has Jacobian $A$. Moreover $A \cdot \mathbb{R}^n = \mathbb{R}^n$. Hence

$$\int_{\mathbb{R}^n} f(Ax) d\mu(x) = \frac{1}{|\det(A)|} \int_{\mathbb{R}^n} f(Ax)|\det(A)| d\mu(x) = \frac{1}{|\det(A)|} \int_{\mathbb{R}^n} f(x) d\mu(x)$$

Thus

$$\mathrm{mod}_G(A) = \frac{\displaystyle\int_{\mathbb{R}^n} f(Ax)d\mu(x)}{\displaystyle\int_{\mathbb{R}^n} f(x)d\mu(x)} = \frac{1}{|\det(A)|}$$