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Problem sheet 1

1. Let A be the adjacency matrix of a graph $G(V, E)$, X and Y subsets of $V(G)$ with the characteristic vectors $\mathbb{1}_X$ and $\mathbb{1}_Y$, respectively, and $E(X, Y)$ be the set of edges between X and Y. Relate the size of $E(X, Y)$ and the value of $(\mathbb{1}_X)^T A \mathbb{1}_Y$ with each other by appropriate (in)equalities in the cases when:

a) X and Y are disjoint,

b) $X = Y$,

c) any other case.

2. Let G be a d-regular graph with the adjacency matrix A. Show that if the number $-d$ is an eigenvalue of A, then G is a bipartite graph.

3. Prove that G is bipartite if and only if the spectrum of its adjacency matrix A is symmetric, i.e. if λ_i is an eigenvalue then so is $-\lambda_i$ (with the same multiplicity). You may find your own proof or you can follow these steps to show the $\Leftarrow$ direction:

a) G is bipartite if and only if every cycle has even length.

b) The entries $(A^t)_{ij}$ of the matrix A^t correspond to the number of walks of length t connecting the i-th vertex with the j-th vertex. Here, a walk allows to use the same edges multiple times.

c) Conclude that you have to show that $(A^t)_{i,i} = 0$ for all i and odd t.

d) Show that the trace $\operatorname{tr} A^t = \sum_i (A^t)_{i,i}$ of A^t vanishes for all odd t, and then ... :-).

As for the other implication, construct from an eigenvector f of A associated to eigenvalue λ_i a new eigenvector $\tilde{f}$ to eigenvalue $-\lambda_i$.

Please turn over!

4. During the course, we will often construct new graphs from a given graph G. Let $V = V(G)$ denote the set of vertices of G and define a new graph $G^{[t]}$ with the vertex set $V\left(G^{[t]}\right)$ being the t-fold cartesian product V^t, i.e.,

$$V\left(G^{[t]}\right) = \{\underline{v} = (v_1, \dots, v_t) : v_1, \dots, v_t \in V\},$$

and let there be an edge between two vertices $\underline{v}$ and $\underline{w}$, where $\underline{v} = (v_1, \dots, v_t)$ and $\underline{w} = (w_1, \dots, w_t) \in V(G^{[t]})$, if the subgraph of G induced by the vertices $\cup_{i,j}\{v_i\} \cup \{w_j\}$ is a clique in G, i.e. is a complete subgraph of G.

For a graph $G(V, E)$, let $\omega(G)$ denote the maximal cardinality of a clique in G, i.e.,

$$\omega(G) = \max_{H \subseteq V}\{|V(H)| : H \text{ is a clique}\}.$$

a) Prove that $\omega(G^{[t]}) = \omega(G)^t$.

Hint: Think about the two inequalities $\leq$ and $\geq$ separately.

b) Use the previous part to deduce that if there is a polynomial time approximation algorithm A for Max-Clique problem with the approximation factor α, where $0 < \alpha < 1$, then for every β such that $0 < \beta < 1$ there exists a polynomial time algorithm B for Max-Clique problem with the approximation factor β.

5. Calculate the spectrum (of the adjacency matrix) of the graph K_n (the complete graph with n vertices).

For any questions concerning the exercises, you can write an email to Jan Volec, `jan@ucw.cz`.