Dr. Maurice Cochand
Assistant: Jan Volec

Problem sheet 3

1. Show that if there exists a polynomial time algorithm (in $n = \#(V(G))$) computing the clique number $\omega(G)$ for every graph $G(V, E)$, then there exists a polynomial time algorithm outputting a maximal clique in G (i.e. of cardinality $\omega(G)$).

Hint: compare for various $v \in V(G)$ the value of $\omega(G)$ and $\omega(G - v)$, where $G-v$ is the graph obtained from G by removing v and all the edges incident to v.

2. Let $f(n) : \mathbb{N} \to \mathbb{N}$ be an increasing function so that $\lim_{n\to\infty} f(n) = \infty$. Recall the proof of the non-approximability of the Max-Clique problem from the lecture. Show that the construction $G^{\{k\}}$ from Step 3 cannot is not good enough in order to prove that if $P \neq NP$, then there is no polynomial-time approximation algorithm for Max-Clique with approximation ratio $f(|V(H)|)$, where H is the input graph.

Hint: Notice that k has to be a function of $|V(H)|$ in order to satisfy

$$\left(\frac{a}{b}\right)^k \geq f(|V(H)|),$$

where a is the clique number of H if H is the graph constructed from a satisfiable 3-SAT formula, and b is the upper bound on the clique number of H if H is the graph constructed from a 3-SAT formula with not more than r-fraction of satisfiable clauses.

3. Given a constant r from PCP theorem, where $3/4 < r < 1$, and Gabber-Galil construction of expanders (recall that $\overline{\lambda} \leq \left(1+\sqrt{3}\right)/3 \sim 0.91$), evaluate the parameters of the construction $\Box G^{\{k\}}$. Specifically, what is a lower (upper) bound on its clique number in the case G comes from a satisfiable formula (a formula with less than r-fraction of satisfiable clauses)?

4. Consider the graph G with the (uncountable!) vertex set $\mathbb{R}^2$ and define the edges by the maps $S(x, y) = (x, x + y)$ and $T(x, y) = (x + y, y)$ so that a vertex $(x, y) \in \mathbb{R}^2$ is adjacent to the points $S(x, y)$, $S^{-1}(x, y)$, $T(x, y)$ and $T^{-1}(x, y)$

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(similarly to the Gabber-Galil construction). This is a 4-regular graph and we look at the following *continuous* vertex-expansion rate $\mu(A)$ of a compact subset $A \subset \mathbb{R}^2$ with a positive measure:

$$\mu(A) = \frac{|S(A) \cup S^{-1}(A) \cup T(A) \cup T^{-1}(A) \cup A|}{|A|},$$

where $|B|$ denotes the Lebesgue measure of a set $B \subseteq \mathbb{R}^2$. We want to prove that for A as above one has $\mu(A) \geq 2$.

The idea of the proof is to reduce the general problem to the case where A is invariant under reflection through both coordinate axes and the diagonal axis, that is, A agrees with the following sets:

$$-A = \{(-x, y) \mid (x, y) \in A\} \qquad A^T = \{(y, x) \mid (x, y) \in A\}$$
$$A^c = \{(-y, x) \mid (x, y) \in A\}.$$

For notational convenience, define $U(A) = S(A) \cup S^{-1}(A) \cup T(A) \cup T^{-1}(A) \cup A$.

a) Prove that if two sets A and $\overline{A}$ satisfy

(1) $|A| = |\overline{A}|$ (2) $|U(A \cap \overline{A}) \cap U(A \backslash \overline{A})| = |U(A \cap \overline{A}) \cap U(\overline{A} \backslash A)|$

(3) $|U(A)| = |U(\overline{A})|$ (4) $|U(A \cap \overline{A})| = |U(\overline{A} \backslash A)|$

then $\mu(A \cup \overline{A}) \leq \mu(A)$ or $\mu(A \cap \overline{A}) \leq \mu(A)$.

Hint: Consider the two cases $A \cap \overline{A} = \varnothing$ and $A \cap \overline{A} \neq \varnothing$ separately.

b) Show that if $\overline{A} \in \{-A, A^T, A^C\}$, then $\overline{A}$ satisfies all the properties $(1) - (4)$.

c) Deduce that we may assume that $A = -A = A^T = A^C$.

d) Show that $\mu(A) \geq 2$ for such a symmetric set. For that, consider the sets $A_i = A \cap Q_i$, where Q_i denotes the ith quadrant, all with equal measure $|A|/4$ (why is that, btw?). Show that the images A_1 of under S and T are disjoint subsets of Q_1 (and similar relations for A_2, A_3, A_4) and prove that $|U(A)| \geq 8(|A|/4)$.

For any questions concerning the exercises, you can write an email to Jan Volec, `jan@ucw.cz`.