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Solution 1

1. a) We calculate that

$$\mathbb{1}_S^T A \mathbb{1}_T = \sum_{i\in S, j\in T} A_{ij} = \sum_{(i,j)\in E\cap S\times T} 1$$

which equals the number of pairs (i,j) that connect S to T. If S and T are disjoint, then if (i,j) is counted implies that (j,i) will not appear in the sum, so that we're safe not to count any edges twice. In particular, the above sum equals $\#(E(S,T))$. To get some feeling for notation, we do a more pedestrian calculation in the next part where we assuming that $S = T$.

b) Let us begin with some observations: It holds $E(V,V) = E$ and there is a edge between the ith and jth vertex if $A_{ij} = 1$ so that $\sum_{ij} A_{ij}$ will count every edge twice since the pair (i,j) is equivalent to (j,i) in an undirected graph (this needs the assumption that we have no loops, i.e. $a_{ii} = 0$). Thus $2\#(E(V,V)) = \sum A_{i,j} = \mathbb{1}^T A \mathbb{1}$ where $\mathbb{1} = \mathbb{1}_V$ is just the everywhere 1 vector. In particular, if we restrict to the subgraph induced by S, the edge set is $S \times S \cap E$ and the corresponding adjacency matrix can be written as $P_S A P_S$ [1], where the projection P_S satisfies $(P_S)_{ij} = 1$ iff $i = j \in S$ and is zero everywhere else and thus

$$2\#(E(S,S)) = \sum_{i,j} (P_S A P_S)_{ij} = \sum_{i,l,k,j} P_{Sik} A_{kl} P_{Slj} = \sum_{j,i\in S} A_{i,j} = \sum_{i,j} (\mathbb{1}_S)_i A_{ij} (\mathbb{1}_S)_j$$

which is $\mathbb{1}_S^T A \mathbb{1}_S$.

c) From the two calculations above, we may deduce that for any two vertices i and j that lie in the intersection $T \cap S$, the edge (i,j) in the sum $\mathbb{1}_S^T A \mathbb{1}_T$ (first formula in a)) is counted twice. Thus $\#(E(S,T)) \le \mathbb{1}_S^T A \mathbb{1}_T \le 2\#(E(S,T))$

[1] One P_S appearing can be interpreted as row restricted to S and the other as column restriction, or equivalently we first restrict the starting vertex of an edge to lie in S and then the end vertex to lie in S.

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2. There was a typo in the assignment, the statement holds only when G is connected. Suppose by contradiction that x is an eigenvector associated to eigenvalue $-d$, and that $x_j = 1$ is its maximal component in absolute value. Then similar in spirit to a) and b),

$$-d = -dx_j = (Ax)_j = \sum_{v_i \in N(v_j)} x_i \geq - \sum_{v_i \in N(v_j)} 1 = -d.$$

This forces every x_i to be equal to -1 where v_i is a neighbor of v_j. Continuing, replacing x_j with x_i, we will walk through every vertex v_j in the graph and thus have that

$$x_i = \begin{cases} 1 & d(v_i, v_j) \text{ is even} \\ -1 & d(v_i, v_j) \text{ is odd} \end{cases}$$

This gives rise to a partition of G, $W = \{v_i : x_i = 1\}$ and $W = \{v_i : x_i = -1\}$ (compare to exercise 2a) so that the graph must be bipartite. Contradiction.

3. We will follow the steps given on the sheet. Let us assume the graph is connected for we can do the analysis with every connected component separately. A cycle is a closed walk, repetition of nodes and edges is possible.

a) By definition, the graph is bipartite if we can partition it into two vertices sets $W, W^c \subset V(G)$ such that there is no edge between any two x, y lying in the same partition elements. This implies that for any edge (say from W to W^c) in a cycle there is a must be follow-up edge (connecting W^c to W again, so that the length is always even.

Now assume that there're no odd cycles. Let u be any fixed vertex and define the sets $W = \{v : d(v, u) \text{ is even}\}$ and $W^c = \{v : d(v, u) \text{ is odd}\}$ where $d(u, v)$ denotes the length of the shortest path between u and v. Assume that two vertices x and y in W share a common edge, $\{\, x, y \,|\, \in \}E(G)$. Consider two shortest path connecting x and u, and y and u respectively. Concatenate them and add the edge $\{\, x, y \,|\, \}$ to get an odd cycle as both path have by assumption even length and the additional edge will flip the parity. This contradicts that there're no odd cycles.(There's a slight inaccuracy here: if the two path will have a common vertex before u, say v, we will have to turn-around already at this vertex in order to have no repeated vertices. This, however, will not change the parity, as the cut-off sections connecting u and v respectively will have the same parity (and in fact have the same length), so that also the new sections connecting x with v and y with v will have the same parity.

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b) We calculate

$$(A^2)_{ij} = \sum_k A_{ik}A_{kj}$$
$$(A^3)_{ij} = \sum_{k_1,k_2} A_{ik_1}A_{k_1k_2}A_{k_2j}$$
$$\vdots$$
$$(A^t)_{ij} = \sum_{k_1,\ldots,k_{t-1}} A_{ik_1}\ldots A_{k_{t-1}j}.$$

Every non-vanishing summand $A_{ik_1}\ldots A_{k_{t-1}j}$ corresponds to a walk of length t from the ith vertex to the jth, and as all possible combinations appear here, the number $(A^t)_{ij}$ gives exactly the number of those walks.

c) Note that there can only be a walk connecting the same vertex with itself if there is also a loop at this vertex, hence there are no odd-length cycles if and only if $(A^t)_{ii} = 0$ for all i and odd t.

d) Since $\sigma(A)^t = \sigma(A^t)$ (the spectral calculus in its simplest form) and the symmetry assumpion on $\sigma(A)$ one finds that

$$\operatorname{tr} A^t = \sum_{\lambda_i\in\sigma(A)} \lambda_i^t = \sum_{0\leq\lambda_i\in\sigma(A)} (\lambda_i^t + (-\lambda_i)^t) = 0$$

for odd t, where the sums run over the same eigenvalue with its corresponding multiplicity. The entries $(A^t)_{ii}$ on the other hand are positive, so that $\operatorname{tr} A^t = \sum(A^t)_{ii}$ can only vanish if in fact all the summands vanish. This concludes the $\Rightarrow$ direction.

To show the other direction, assume that G is bipartite with partition W and W^c as stated in a). List first all the vertices $\{v_1,\ldots,v_{|W|}\}$ in W and then the other vertices $\{v_{|W|+1},\ldots,v_{|V|}\}$ in W^c so that the corresponding adjacency matrix is in the block form

$$A = \left[\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array}\right].$$

From this representation it is easy to see that if $x = (w, w^c)$ is an eigenvector to eigenvalue λ_k then $\tilde{x} = (w, -w^c)$ is an eigenvector to eigenvalue $-\lambda_k$: The equation $Ax = (Bw^c, B^Tw) = \lambda(w, w^c)$ implies that $A\tilde{x} = (B-(w^c), B^Tw) = \lambda(-w, w^c) = -\lambda(w, -w^c) = -\lambda\tilde{x}$. This will give the right multiplicity as well since if we take a set of eigenvectors $\{x\}$ to be linearly independent, then the $\{\tilde{x}\}$ will be linearly independent as well.

4. As noted we will show the two inequalities separately. First assume that H is a clique in G. This clique will give rise $\#(H)^t$ vertices in $G^{[t]}$, each of the

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form $(v_1, \ldots, v_t)$ with $v_i \in H$. As any subgraph of a complete subgraph is again complete, any two of these vertices will be connected in $G^{[t]}$ so that we find a clique in $G^{[t]}$ of size $\#(H)^t$. In particular, $\omega(G^{[t]}) \geq \omega(G)^t$.

Now assume that $\tilde{H}$ is a clique in $G^{[t]}$ and let denote $H \subset G$ the subgraph induced by all the vertices that appear as a component in some vertex in $\tilde{H}$. We claim that $\#(H)^t \geq \#(\tilde{H})$. But $\#(H)^t$ is the number of all possible combinations of vertices in G^t constructed from the vertices of H. By assumption of H, $\tilde{H}$ is a subset of these, and so the last inequality follows. We finally note that since $\tilde{H}$ is a clique and by definition of $G^{[t]}$, H is in fact also a clique in G and thus $\omega(G^{[t]}) \leq \omega(G)^t$.