Problem set – Week 1

Linear systems & Gauss elimination

1. Solve the following linear systems via elimination.

$$(a)\ \left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 8 \\ x & + & 3y & + & 3z & = & 10 \\ x & + & 2y & + & 4z & = & 9 \end{array}\right| \qquad (b)\ \left|\begin{array}{ccccc} x & - & 2y & = & 2 \\ 3x & + & 5y & = & 17 \end{array}\right|$$

$$(c)\ \left|\begin{array}{ccccccc} x & + & 4y & + & z & = & 0 \\ 4x & + & 13y & + & 7z & = & 0 \\ 7x & + & 22y & + & 13z & = & 0 \end{array}\right| \qquad (d)\ \left|\begin{array}{ccccccc} x & + & 4y & + & z & = & 0 \\ 4x & + & 13y & + & 7z & = & 0 \\ 7x & + & 22y & + & 13z & = & 1 \end{array}\right|$$

Sketch the solutions of (b) graphically, as intersection of lines in the x-y-plane. Describe your solutions to (c) in terms of intersecting planes. Here are another two linear systems to solve.

$$(e)\ \left|\begin{array}{ccccc} x & + & y & = & 1 \\ 2x & - & y & = & 5 \\ 3x & + & 4y & = & 2 \end{array}\right| \qquad (f)\ \left|\begin{array}{ccccccccc} x_1 & + & & & 2x_3 & + & 4x_4 & = & -8 \\ & & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ 3x_1 & + & 4x_2 & - & 6x_3 & + & 8x_4 & = & 0 \\ & - & x_2 & + & 3x_3 & + & 4x_4 & = & -12 \end{array}\right|$$

2. Consider the linear system

$$\left|\begin{array}{ccccccc} x & + & y & - & z & = & -2 \\ 3x & - & 5y & + & 13z & = & 18 \\ x & - & 2y & + & 5z & = & k \end{array}\right|$$

where k is an arbitrary constant.

(a) For which value(s) of k does this system have one or infinitely many solutions ?

(b) For each of these values, how many solutions does the system have ?

(c) Write down all solutions.

3. Why are linear systems particularly easy to solve when they are in triangular form ? Answer by considering the upper triangular system

$$\left|\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & x_3 & + & 4x_4 & = & -3 \\ & & x_2 & + & 3x_3 & + & 7x_4 & = & 5 \\ & & & & x_3 & + & 2x_4 & = & 2 \\ & & & & & & x_4 & = & 0 \end{array}\right|$$

4. Find a system of linear equations with three unknowns whose solutions are the points on the line through $(1,1,1)$ and $(3,5,0)$.

5. We call a function f a polynomial of degree 2 if it is of the form $f(t) = at^2 + bt + c$, with $a \neq 0$. Find the polynomial of degree 2 whose graph passes through the points $(-1, 1)$, $(2, 3)$ and $(3, 13)$ in the x-y-plane.

6. Let us assume that parking meters in Zurich only accept coins of 20ct, 50ct and 1 Fr. As an incentive, the city council offers a reward to any patrolman who, from his daily round, brings back exactly 1000 coins, worth exactly 1000 Fr.

   What are the odds for this reward to be claimed any time soon ?