Problem set – Week 13

Linear differential equations

1. Find the solution of
$$y' = 5x - \frac{3y}{x}$$
with initial condition $y(1) = 2$.

2. Find the solution for
   (a) $y'' + 4y = 0$
   (b) $2y'' + 7y' = 4y$
   (c) $y'' + 2y' + y = 0$
   (d) $y''' - y'' - 9y' + 9y = 0$

3. Find the solution for $y'' = x + y$.

4. The goal of this exercise is to prove that any initial value problem for the simultaneous system of differential equations
$$\left| \begin{array}{rcl} dx/dt &=& y + x(1 - x^2 - y^2) \\ dy/dt &=& -x + y(1 - x^2 - y^2) \end{array} \right|$$
has a uniquely determined solution $(x(t), y(t))$ for all $t \in \mathbb{R}$.
   (a) The system above probably has a simpler form if expressed in polar coordinates. Set $x = r\cos\theta$, $y = r\sin\theta$ and rewrite the system as
$$\left| \begin{array}{rcl} dr/dt &=& \dots \\ d\theta/dt &=& \dots \end{array} \right|$$
   (b) Solve the system you found in (a).
   (c) Check that if the initial condition on the system is $(x(0), y(0)) = (0, 0)$, the unique solution is $(x(t), y(t)) = (0, 0)$ for all $t \in \mathbb{R}$.
   (d) Suppose the initial condition is $(x(0), y(0)) \neq (0, 0)$. Find the solution $(x(t), y(t))$ of the initial value problem.
   (e) Observe that the solution you determined in (d) is uniquely determined and exists for all $t \in \mathbb{R}$.