Problem set – Week 7

Eigenvalues and eigenvectors
Changing bases

1. For which values of k does $\begin{pmatrix} -1 & k \\ 4 & 3 \end{pmatrix}$ have 5 as an eigenvalue ?

2. Check that the characteristic polynomial of a 2×2-matrix A is

$$p_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A).$$

   Let λ_1 and λ_2 be eigenvalues of A. What is the relation between $\mathrm{tr}(A)$, $\det(A)$, λ_1 and λ_2 ?

   If you are stuck, consider explicitly $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

3. Determine all eigenvalues and eigenvectors of $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

4. A reflection about a line in the plane passing through the origin is expressed via a matrix of the form

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

   with $a^2 + b^2 = 1$. Conversely any such matrix represents a reflection about a line. Express the matrix

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

   as a reflection combined with a scaling.

   Determine geometrically, i.e. without computing the characteristic equation, all the eigenvalues of A, and give two linearly independent eigenvectors.

5. Let A be a symmetric $n \times n$ matrix. Prove the following statements.

   (a) $A\vec{v} \cdot \vec{w} = \vec{v} \cdot A\vec{w}$.

   (b) If $\vec{v}$ and $\vec{w}$ are two eigenvectors of A with distinct eigenvalues, then $\vec{w}$ is orthogonal to $\vec{v}$.

   (c) Find a basis of $\mathbb{R}^3$ that consists of eigenvectors for

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

6. For the following data, find the matrix B of the linear transformation $T\vec{v} = A\vec{v}$ with respect to the basis $\mathcal{B}$. For practice, solve each problem in three ways ; using the formula $B = S^{-1}AS$, using a commutative diagram, constructing B column by column.

   (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$

   (b) $A = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$