Problem set – Week 12

Linear differential equations

1. Find the solution of
$$y' = 5x - \frac{3y}{x}$$
with initial condition $y(1) = 2$.
Solution : $y = x^2 + x^{-3}$

2. Find the solution for
   (a) $y'' + 4y = 0$
   Solution : $y = c_1 \cos(2x) + c_2 \sin(2x)$
   (b) $2y'' + 7y' = 4y$
   Solution : $y = c_1 e^{x/2} + c_2 e^{-4x}$
   (c) $y'' + 2y' + y = 0$
   Solution : $y = c_1 e^{-x} + c_2 x e^{-x}$
   (d) $y''' - y'' - 9y' + 9y = 0$
   Solution : $y = c_1 e^{-3x} + c_2 e^{3x} + c_3 e^{x}$

3. Find the solution for $y'' = x + y$.
Solution : $y = c_1 e^{x} + c_2 e^{-x} - x$

4. The goal of this exercise is to prove that any initial value problem for the simultaneous system of differential equations
$$\left| \begin{array}{rcl} dx/dt & = & y + x(1 - x^2 - y^2) \\ \\ dy/dt & = & -x + y(1 - x^2 - y^2) \end{array} \right|$$
has a uniquely determined solution $(x(t), y(t))$ for all $t \in \mathbb{R}$.
   (a) The system above probably has a simpler form if expressed in polar coordinates. Set $x = r\cos\theta$, $y = r\sin\theta$ and rewrite the system as
$$\left| \begin{array}{rcl} dr/dt & = & \dots \\ d\theta/dt & = & \dots \end{array} \right|$$
   Solution :
$$\left| \begin{array}{rcl} dr/dt & = & r(1 - r^2) \\ d\theta/dt & = & -1 \end{array} \right|$$
   (b) Solve the system you found in (a).
   Solution : $(r(t), \theta(t)) = \left(\frac{c_1 e^t}{\sqrt{1 + c_1^2 e^{2t}}}, -t + c_2 \right)$

(c) Check that if the initial condition on the system is $(x(0), y(0)) = (0, 0)$, the unique solution is $(x(t), y(t)) = (0, 0)$ for all $t \in \mathbb{R}$.

Solution : Clear, since $dx/dt = dy/dt = 0$.

(d) Suppose the initial condition is $(x(0), y(0)) \neq (0, 0)$. Find the solution $(x(t), y(t))$ of the initial value problem.

Solution : Set $(x(0), y(0)) =: (x_0, y_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0)$. Then $c_1 = \frac{r_0}{\sqrt{1-r_0^2}}$, $c_2 = \theta_0$. Then $(x(t), y(t)) = (r(t)\cos(\theta(t)), r(t)\sin(\theta(t)))$.

(e) Observe that the solution you determined in (d) is uniquely determined and exists for all $t \in \mathbb{R}$.