Solutions – Week 5

Level curves, level surfaces, partial derivatives, vector fields and line integrals

1. For each of the following functions, sketch the surface $z = f(x, y)$ and a typical level curve.

   (a) $f(x, y) = y^2$

   (b) $f(x, y) = 1 - |x| - |y|$

   (c) $f(x, y) = \sqrt{x^2 + y^2 - 4}$

   For each of the following functions, sketch a typical level surface

   (d) $f(x, y, z) = \ln(x^2 + y^2 + z^2)$

   (e) $f(x, y, z) = z$

   (f) $f(x, y, z) = x^2 + y^2$

   Solutions : Level curves : (a) the two parallel lines $y = \sqrt{c}$ and $y = -\sqrt{c}$; (b) lozenge centered at the origin ; (c) circle centered at the origin ; (d) sphere centered at the origin ; (e) plane parallel to xy-plane ; (f) cylinder centered around z-axis.

2. Find the line that is tangent to the intersection of $z = \arctan(xy)$ with the plane $x = 2$ at $(2, 1/2, \pi/4)$.

   Solution : $(1, 1/2, \pi/4) + t(0, 1, 1)$

3. We know that if f_{xy} and f_{yx} exist, then $f_{xy} = f_{yx}$. For the following functions, try to determine what is the fastest way of computing f_{xy} without writing anything down ; should you differentiate in x or in y first ?

   (a) $f(x, y) = x \sin y + e^y$

   (b) $f(x, y) = 1/x$

   (c) $f(x, y) = y + (x/y)$

   (d) $f(x, y) = y + x^2 y + 4y^3 - \ln(y^2 + 1)$

   (e) $f(x, y) = x^2 + 5xy + \sin x + 7e^x$

   (f) $f(x, y) = x \ln xy$

   Solutions : (a) x ; (b) y ; (c) x ; (d) x ; (e) y ; (f) y.

4. Compute the mass of a wire that lies along the curve $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}$, $t \in [0, 1]$, if the density is $\delta = (3/2)t$.

   Solution : $2\sqrt{2} - 1$.

5. Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field that points toward the origin with magnitude inversely proportional to the square of the distance from (x, y) to $(0, 0)$. Note : The field is not defined at the origin.

   Solution : $\mathbf{F}(x, y) = \frac{-x}{(x^2+y^2)^{3/2}}\mathbf{i} + \frac{-y}{(x^2+y^2)^{3/2}}\mathbf{j}$.

6. Determine the work done by the gradient of $f(x, y) = (x + y)^2$ counterclockwise around the circle $x^2 + y^2 = 4$ from $(2, 0)$ to itself.

   Solution : 0.

7. A particle is moving along a smooth curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$. The force moving the particle has constant magnitude k and is always pointing away from the origin. Show that the work done by the force is

$$k\left((b^2 + (f(b))^2)^{1/2} - (a^2 + (f(a))^2)^{1/2}\right).$$

   Solution : To be discussed in class.