Problem set – Week 6

Chain rule for partial derivatives change of bases and determinants

1. The lengths a, b, c of the edges of a rectangular box are changing with time. At some fixed time t_0, we have the data $a = 1$, $b = 2$, $c = 3$, $da/dt = db/dt = 1$ and $dc/dt = -3$.

   At what rates are the volume and the surface area of the box changing at t_0 ? Are the interior diagonals of the box increasing or decreasing in length ?

   Solutions : Volume $V'(t_0) = 3$, surface area $A'(t_0) = 0$, diagonals are decreasing in length at t_0.

2. Under mild continuity restrictions, if $F(x) = \int_a^b g(t, x)dt$, then $F'(x) = \int_a^b g_x(t, x)dt$. Together with the chain rule, this allows to derivate $F(x) = \int_a^{f(x)} g(t, x)dt$ by considering $G(u, x) = \int_a^u g(t, x)dt$ where $u = f(x)$.

   Find the derivative of

$$F(x) \;=\; \int_0^{x^2} \left(t^2 + x^3\right)^{3/2} dt.$$

   Solution : $2x^{11/2}(x + 1)^{3/2} + \frac{9}{4}x^{9/2}\sqrt{x + 1} + \frac{9}{4}x^5 \ln\left|\sqrt{x + 1} + \sqrt{x}\right|$.

3. For the following sets of data, find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to basis $\mathcal{B}$.

   (a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$;

   Solution : $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

   (b) $T(\vec{x}) = \vec{v}_2 \times \vec{x}$ and $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis of perpendicular vectors, such that $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$.

   Solution : $B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

4. Show that if the 3×3 matrix A represents the reflection about a plane, then A is similar to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

   Solution : To be discussed in class.

5. Find the derivative of

$$f(x) = \det \begin{pmatrix} 1 & 1 & 2 & 3 & 4 \\ 9 & 0 & 2 & 3 & 4 \\ 9 & 0 & 0 & 3 & 4 \\ x & 1 & 2 & 9 & 1 \\ 7 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Solution : $f'(x) = -3$.

6. Compute the inverse of $\begin{pmatrix} 0 & -1 & \alpha \\ -1 & 2 & 0 \\ -2 & 4 & 1 \end{pmatrix}$, where $\alpha \in \mathbb{R}$. How does the inverse depends on α ?

Solution :

$$A^{-1} = \begin{pmatrix} -2 & -1-4\alpha & 2\alpha \\ -1 & -2\alpha & \alpha \\ 0 & -2 & 1 \end{pmatrix}.$$

7. Consider the following $n \times n$ matrix,

$$T_n := \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & 0 & & \vdots \\ 0 & 1 & 2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2 \end{pmatrix}.$$

(a) Using the Laplace expansion, prove that

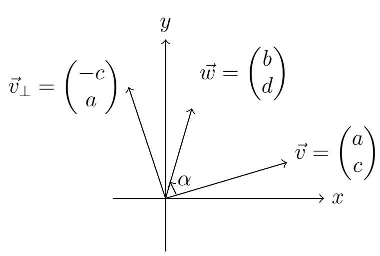
$$\det(T_n) = 2 \cdot \det(T_{n-1}) - \det(T_{n-2}) \quad \text{for} \quad n > 2.$$

(b) Prove by induction that

$$\det(T_n) = 1 \quad \text{for all } n \in \mathbb{N}.$$

Solution : To be discussed in class.

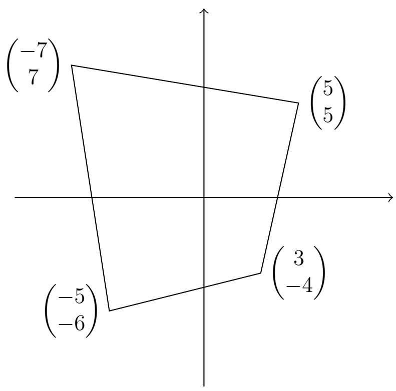
8. Verify



$$ad - bc \;=\; \vec{v}_\perp \cdot \vec{w} \;=\; ||\vec{v}_\perp||\,||\vec{w}||\cos\left(\frac{\pi}{2} - \alpha\right) \;=\; ||\vec{v}||\,||\vec{w}||\sin\alpha.$$

What is the geometric interpretation of $|\det(\vec{v}|\vec{w})|$ and how does $\det(\vec{v}|\vec{w})$ depend on $\alpha = \measuredangle(\vec{v}, \vec{w})$?

Use your answer to find the area of the following region :



Solution : $|\det(\vec{v} \mid \vec{w})|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$. The area of the region is 75.