Problem set – Week 7

Eigenvalues and eigenvectors
Changing bases

1. For which values of k does $\begin{pmatrix} -1 & k \\ 4 & 3 \end{pmatrix}$ have 5 as an eigenvalue ?

   Solution : $k = 3$.

2. Check that the characteristic polynomial of a 2×2-matrix A is

   $$p_A(\lambda) \;=\; \lambda^2 - \operatorname{tr}(A)\lambda + \det(A).$$

   Let λ_1 and λ_2 be eigenvalues of A. What is the relation between $\operatorname{tr}(A)$, $\det(A)$, λ_1 and λ_2 ?

   If you are stuck, consider explicitly $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

   Solution : $\lambda_1 + \lambda_2 = \operatorname{tr}(A)$, $\lambda_1\lambda_2 = \det(A)$.

3. Determine all eigenvalues and eigenvectors of $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

   Solution : $\lambda_1 = \sqrt{3}$, $\lambda_2 = -\sqrt{3}$, $\lambda_3 = 3$, $v_1 = \begin{pmatrix} -1 \\ 2+\sqrt{3} \\ -1-\sqrt{3} \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 2-\sqrt{3} \\ -1+\sqrt{3} \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

4. A reflection about a line in the plane passing through the origin is expressed via a matrix of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with $a^2 + b^2 = 1$. Conversely any such matrix represents a reflection about a line. Express the matrix $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ as a reflection combined with a scaling.

   Determine geometrically, i.e. without computing the characteristic equation, all the eigenvalues of A, and give two linearly independent eigenvectors.

   Solution : $\lambda = \pm 5$. Take a vector $\vec{v}_1$ on the line and vector $\vec{v}_2$ orthogonal to $\vec{v}_1$.

5. Show that similar matrices have the same eigenvalues. Show that transpose matrices have the same eigenvalues.

   Solution : $S^{-1}AS\vec{v} = \lambda\vec{v} \Leftrightarrow AS\vec{v} = \lambda S\vec{v}$, $\det(A - \lambda I) = \det({}^t(A - \lambda I)) = \det({}^tA - \lambda I)$.

6. Let A be a symmetric $n \times n$ matrix. Prove the following statements.

   (a) $A\vec{v} \cdot \vec{w} = \vec{v} \cdot A\vec{w}$.

   Solution : Compare row by row.

   (b) If $\vec{v}$ and $\vec{w}$ are two eigenvectors of A with distinct eigenvalues, then $\vec{w}$ is orthogonal to $\vec{v}$.

   Solution : $A\vec{v} \cdot \vec{w} = \lambda_1\vec{v} \cdot \vec{w} = \vec{v} \cdot \lambda_2\vec{w} = \lambda_2\vec{v} \cdot \vec{w} \Rightarrow (\lambda_1 - \lambda_2)\vec{v} \cdot \vec{w} = 0 \Rightarrow \vec{v} \cdot \vec{w} = 0$.

   (c) Find a basis of $\mathbb{R}^3$ that consists of eigenvectors for

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

   Solution : $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1+\sqrt{33} \\ 8 \\ 1+\sqrt{33} \end{pmatrix}, \begin{pmatrix} 1-\sqrt{33} \\ 8 \\ 1-\sqrt{33} \end{pmatrix}$.

7. For the following data, find the matrix B of the linear transformation $T\vec{v} = A\vec{v}$ with respect to the basis $\mathcal{B}$. For practice, solve each problem in three ways ; using the formula $B = S^{-1}AS$, using a commutative diagram, constructing B column by column.

   (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$

   Solution : $B = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$.

   (b) $A = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$

   Solution : $B = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}$.