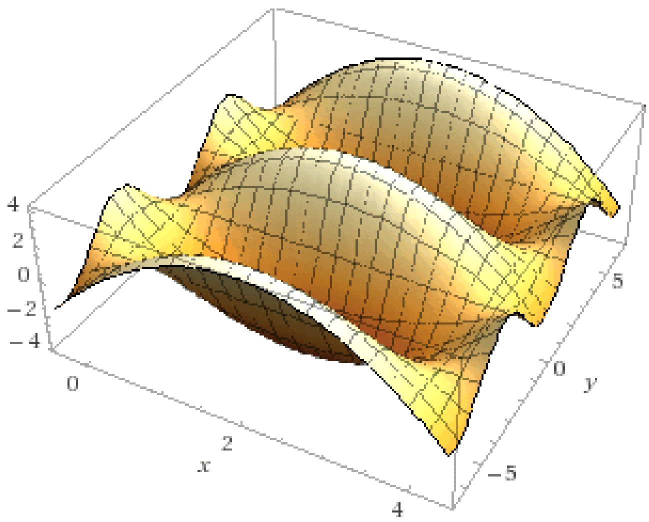
Problem set – Week 8

Extrema problems

1. Find the absolute extrema of the surface $f(x,y) = (4x - x^2)\cos(y)$ on the rectangular plate $1 \le x \le 3$, $-\pi/4 \le y \le \pi/4$.



Solutions : Abs. max is 4 at (2,0), abs. min is $3\sqrt{2}/2$ at points $(3, \pm\pi/4)$, $(1, \pm\pi/4)$.

2. A flat circular plate P of radius 1 is heated (included the boundary of the plate) so that the temperature at the point $(x,y) \in P$ is

$$T(x,y) = x^2 + 2y^2 - x.$$

Find the temperatures at the hottest and coldest points on the plate.

Solutions : Hottest : $9/4$, coldest : $-1/4$.

3. Find the numbers $a \le b$ such that the integral

$$\int_a^b \left(e^{x^2} - 2\right) dx.$$

has its largest value.

Solution : Ha, oups. As you will have understood after some investigation, I meant to ask you for the smallest value. That happens at $(-\sqrt{\ln 2}, \sqrt{\ln 2})$. Sorry.

4. Find three numbers whose sum is 9 and whose sum of squares is a minimum.

Solution : $a = 3, b = 3, c = 3$.

5. Among all the points on the surface $z = 10 - x^2 - y^2$ that lie above the plane $x + 2y + 3z = 0$, find the point farthest from the plane.

   Solution : $(1/6, 1/3, 355/36)$.

6. The Hessian matrix of $f(x, y) = x^2y^2$ at $(0, 0)$ is the zero matrix. Determine whether the function has an extremum or not at the origin by imagining what the surface looks like.

   Solution : The function is trivially zero on the coordinate axes and positive everywhere else.

7. In this exercise, we give a proof that the geometric mean is $\leq$ the arithmetic mean, for any set of n non-negative real numbers, i.e.

$$(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n}. \tag{1}$$

   (a) Explain why the maximum value of $x^2y^2z^2$ on a sphere of radius r centered at the origin is $(r^2/3)^3$.

   Solution : Follows from solving the extremum problem.

   (b) Deduce (1) from (a) for $n = 3$.

   Solution : a_1, a_2, a_3 are non-negative, hence you can write $a_i = (\sqrt{a_i})^2$, for each $i = 1, 2, 3$. Let $r = \sqrt{a_1 + a_2 + a_3}$. Hence now $(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$ lies on the sphere of radius r centered at the origin, and by part (a),

$$a_1a_2a_3 \leq \left(\frac{a_1 + a_2 + a_3}{3}\right)^3.$$

   (c) Explain how the argument can be seen to hold more generally for any $n \geq 1$.

   Solution : In (a), you will have observed that there are two types of critical points $(x, y, z) \in \mathbb{R}^3$; those with at least one component $= 0$ and the point $(r^2/3, r^2/3, r^2/3)$. The same thing holds in $\mathbb{R}^n$; Fix the n-th coordinate and set

$$f(x_1, \ldots, x_{n-1}) = \prod_{i=1}^{n-1} x_i^2 \left(r^2 - \sum_{i=1}^{n-1} x_i^2\right),$$

   then

$$f_{x_i}(x_1, \ldots, x_{n-1}) = 2x_i \prod_{\substack{j=1 \\ j \neq i}}^{n-1} x_j^2 \left(r^2 - \sum_{j=1}^{n-1} x_j^2 - x_i^2\right).$$

   Take $\vec{x} \in \mathbb{R}^{n-1}$ such that $x_j \neq 0$ for all $j = 1, \ldots, n-1$, then adding up all equations of the linear system

$$\left|\begin{array}{ccc} f_{x_1}(\vec{x}) & = & 0 \\ f_{x_2}(\vec{x}) & = & 0 \\ \vdots & \vdots & \vdots \\ f_{x_{n-1}}(\vec{x}) & = & 0 \end{array}\right|$$

we obtain

$$\sum_{j=1}^{n-1} x_j^2 + (n-1)\sum_{j=1}^{n-1} x_j^2 = n\sum_{j=1}^{n-1} x_j^2 = n(r^2 - x_n^2) = (n-1)r^2$$

hence $x_n^2 = r^2/n$. More generally, fixing the k-th coordinate, you obtain $x_k^2 = r^2/n$. Therefore, the only non-trivial critical point of $x_1^2 \cdots x_n^2$ on the $(n-1)$-dimensional sphere of radius r is $(r^2/n, \ldots, r^2/n)$. Part (b) extends immediately to general n.