Lösung Serie 4

zu Abschnitt 5.2

14.) $-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}$ als Reihe zu schreiben

$$\Rightarrow \sum_{n=1}^{5} (-1)^n \cdot \frac{n}{5}$$

zu Abschnitt 5.3.

6.) $\int_1^2 f(x)\,dx = 5$

a) $\int_1^2 f(u)\,du = 5$

b) $\int_1^2 \sqrt{3} f(z)\,dz = \sqrt{3}\int_1^2 f(z)\,dz = \sqrt{3}\cdot 5$

c) $\int_2^1 f(t)\,dt = -\int_1^2 f(t)\,dt = -5$

d) $\int_1^2 -f(x)\,dx = -\int_1^2 f(x)\,dx = -5$

zu Abschnitt 5.4.

28.) $A=\int_{\pi/6}^{5\pi/6}\left(\sin x - \sin\frac{\pi}{6}\right)dx = \left[-\cos x - 0{,}5x\right]_{\pi/6}^{5\pi/6} = \sqrt{3}-\pi/3$

A5.4/13 START $\sin(2x) = 2\sin(x)\cos(x)$ (Formelsammlung)

$$\int_{\pi/2}^{\pi} \frac{\sin(2x)}{2\sin(x)}\,dx = \int_{\frac{\pi}{2}}^{\pi} \frac{2\sin(x)\cos(x)}{2\sin(x)}\,dx = \int_{\frac{\pi}{2}}^{\pi} \cos(x)\,dx = \Big[\sin(x)\Big]_{x=\frac{\pi}{2}}^{x=\pi} =$$

$$= \sin(\pi) - \sin\left(\frac{\pi}{2}\right) = 0 - 1 = \underline{\underline{-1}}$$

EX

A5.4/16 $F(x)$

a) "Bestimmen Sie die Ableitungen durch Berechnung der Integrals & Diff. des Ergebnis"

$$\frac{d}{dx}\int_1^{\sin(x)} 3t^2\,dt = \frac{d}{dx}\Big[t^3\Big]_{t=1}^{t=\sin(x)} = \frac{d}{dx}\Big[\sin(x)^3 - 1^3\Big] =$$

$$= \cos(x)\cdot 3\sin(x)^2 = \underline{\underline{3\cos(x)\sin(x)^2}}$$

b) "Bestimmen Sie die Ableitungen durch direkte Differentiation des Integrals"

Bekannt: $f(t) = 3t^2 \Rightarrow \frac{d}{dx}\int_1^x f(t)\,dt = F(x) - F(1)$

$$\Rightarrow \frac{d}{dx}\int_1^x f(t)\,dt = F'(x) - 0 = f(x)$$

Jetzt:

$$\int_1^{\sin(x)} f(t)\,dt = F(\sin(x)) - F(1)$$

Kettenregel

$$\Rightarrow \frac{d}{dx}\int_1^{\sin(x)} f(t)\,dt = \frac{d}{dx}\big(F(\sin(x)) - 1\big) = \cos(x)\cdot F'(\sin(x)) = \cos(x) f(\sin(x))$$

$$= \cos(x)\cdot 3\sin(x)^2 =$$

$$= \underline{\underline{3\cos(x)\sin(x)^2}}$$

$\boxed{\text{THEORIE SUB.}}$

A5.4/22 $F(x)$

$$y = \int_0^{\sin(x)} \frac{dt}{\sqrt{1-t^2}}, \quad |x| < \frac{\pi}{2} \qquad \frac{dy}{dx} = ?$$

~~Annahme~~ wir tun so, als ob wir Stammfkt'n kennen

$$f(t) = \frac{1}{\sqrt{1-t^2}}, \quad \frac{d}{dx}\int_0^{\sin(x)} f(t)\,dt = \frac{d}{dx}\Big[F(\sin(x)) - F(0)\Big] = \cos(x)F'(\sin(x)) =$$

$$= \cos(x) f(\sin(x)) = \cos(x)\frac{1}{\sqrt{1-\sin(x)^2}} = \frac{\cos(x)}{\sqrt{\cos^2(x)}} = \underline{\underline{1}}$$

$\sin^2 + \cos^2 = 1$

A5.5/13 SUBSTITUTION

$$\int \sin^5\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right)dx = \int u^5\cdot\cos\left(\frac{x}{3}\right)\cdot\frac{3}{\cos\left(\frac{x}{3}\right)}\,du = \int 3u^5\,du = \frac{1}{2}u^6 = \underline{\underline{\frac{1}{2}\sin\left(\frac{x}{3}\right)^6}}$$

SUB: $u(x) = \sin\left(\frac{x}{3}\right) \Rightarrow \frac{du}{dx} = \frac{1}{3}\cos\left(\frac{x}{3}\right) \Rightarrow dx = \frac{3}{\cos\left(\frac{x}{3}\right)}\,du$

A5.5/31 ANFANGSWERTPROBLEME

$y'(x) = f'(x) = 4x(x^2+8)^{-1/3}, \quad y(0) = 0, \quad \Rightarrow y(x) = ?$

$$y(x) = \int 4x(x^2+8)^{-1/3}\,dx \overset{SUB}{=} \int 4x(u)^{-1/3}\cdot\frac{du}{2x} = \int 2u^{-1/3}\,du = 3u^{2/3} + C$$

$\Rightarrow y(x) = 3(x^2+8)^{2/3} + C$ mit Nebenbedingung $y(0) = 0$

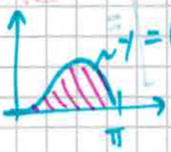
$\Rightarrow y(0) = 3(8^{2/3}) + C \overset{!}{=} 0$ mit $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$

$\Rightarrow 3 \cdot 4 + C \overset{!}{=} 0$

$\Rightarrow C = -12$

$\Rightarrow \underline{\underline{y(x) = 3(x^2+8)^{2/3} - 12}}$

AS. 6/14



1 Grenzen anpassen — Satz 5.7 — $u(\pi) = 1 - \cos(\pi) = 2$

$$\Rightarrow \int_0^\pi (1-\cos(x))\sin(x)\,dx = \int_{u(0)=0}^{u(\pi)} u \sin(x) \frac{du}{\sin(x)} =$$

SVB: $u(x) = 1 - \cos(x) \Rightarrow \frac{du}{dx} = \sin(x) \Rightarrow dx = \frac{du}{\sin(x)}$

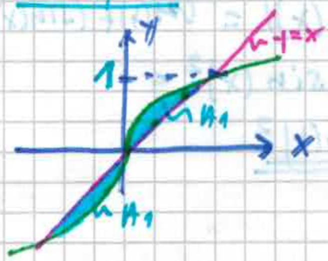
$$= \int_0^2 u\,du = \left[\frac{1}{2}u^2\right]_{u=0}^{u=2} = \frac{1}{2}2^2 - 0 = \frac{4}{2} = \underline{\underline{2}}$$

AS. 6/24

$$A = \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx = \left[\frac{1}{3}x^3\right]_{x=0}^{x=1} + \left[2x - \frac{1}{2}x^2\right]_{x=1}^{x=2} = \frac{1}{3} + [(4-2) - (2-\frac{1}{2})] =$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \underline{\underline{5/6}}$$

AS. 6/45



$x - y^3 = 0 \Rightarrow x_1(y) = y^3$ Schnittpunkt: $x_1(y) \overset{!}{=} x_2(y)$

$x - y = 0 \Rightarrow x_2(y) = y$ $\Rightarrow y \overset{!}{=} y^3 \Rightarrow \underline{y_{SP} = 1}$

$A = 2A_1$

$$A_1 = \int_0^1 y\,dy - \int_0^1 y^3\,dy = \left[\frac{1}{2}y^2\right]_{y=0}^{y=1} - \left[\frac{1}{4}y^4\right]_{y=0}^{y=1} =$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \Rightarrow A = 2 \cdot \frac{1}{4} = \underline{\underline{1/2}}$$

AS 8.1/5

1.P.I.: $\int \underset{f}{x^2} \underset{g'}{e^{-x}}\,dx \overset{P.I.}{=} \underset{f}{x^2}\underset{g}{(-e^{-x})} - \underset{\int f' g}{\int 2x(-e^{-x})\,dx} = -x^2 e^{-x} + \overbrace{\int 2x e^{-x}\,dx}^{=:a}$

2.P.I.: $\overbrace{\int \underset{f}{2x} \underset{g'}{e^{-x}}\,dx}^{=:a} = \underset{f}{(2x)}\underset{g}{(-e^{-x})} - \int \underset{f'}{2}\underset{g}{(-e^{-x})}\,dx = -2xe^{-x} + 2\int e^{-x}\,dx = -2xe^{-x} - 2e^{-x}$

$$\Rightarrow \int x^2 e^{-x}\,dx = -x^2 e^{-x} + (-2xe^{-x} - 2e^{-x}) = \underline{\underline{-e^{-x}(x^2 + 2x + 2)}}$$

8.2/3

$$\int \sin^3(x)\,dx = \int \underset{g'}{\sin(x)}\,\underset{f}{\sin^2(x)}\,dx = \underset{f}{\sin^2(x)}\,\underset{g}{(-\cos(x))} - \int \underset{f'}{\cos(x)\,2\sin(x)}\,\underset{g}{(-\cos(x))}\,dx$$

$$\int \sin^3(x)$$ $\boxed{\text{vgl. mit } \int \sin^m x \cos^n x\,dx}$ Rezept Buch S. 598 Empfehlung: mit Substitution

$\Rightarrow m = 3,\ n = 0$ A/a Fall m ungerade

$\Rightarrow$ Wir setzen $m = 2k+1 \Leftrightarrow 3 = 2k+1 \Rightarrow k = 1$

$$\Rightarrow \sin^3(x) = \sin^{2k+1}(x) = \sin^2(x)\sin(x) = (1-\cos^2(x))\sin(x)$$

$$\int \sin^3(x)\,dx = \int \sin^2(x)\sin(x)\,dx \overset{\sin^2+\cos^2=1}{=} \int (1-\cos^2(x))\sin(x)\,dx \overset{SUB}{=}$$

$$u(x) = \cos(x) \Rightarrow \frac{du}{dx} = -\sin(x) \Rightarrow dx = \frac{du}{-\sin(x)}$$

$$= \int (1-u^2)\sin(x)\frac{du}{-\sin(x)} = \int (u^2-1)\,du = \frac{1}{3}u^3 - u = \underline{\underline{\frac{1}{3}\cos^3(x) - \cos(x)}}$$

A8.2/7

$$\int \cos^2(x)\,dx = \int \underset{f}{\cos(x)}\,\underset{g'}{\cos(x)}\,dx = \underset{f}{\cos(x)}\,\underset{g}{\sin(x)} - \int \underset{f'}{(-\sin(x))}\,\underset{g}{\sin(x)}\,dx$$

$$\int \underset{f}{\sin(x)}\,\underset{g'}{\sin(x)} = \underset{f}{\sin(x)}\,\underset{g}{(-\cos(x))} - \int \underset{f'}{\cos(x)}\,\underset{g}{(-\cos(x))}\,dx = -\sin(x)\cos(x) + \int \cos^2(x)\,dx$$

$$\Rightarrow \int \cos^2(x)\,dx = \cos(x)\sin(x) + \int \sin^2(x)\,dx = \cos(x)\sin(x) - \sin(x)\cos(x) + \int \cos^2(x)\,dx$$

$$\int \cos^2(x)\,dx = \int (1-\sin^2(x))\,dx = \int \frac{1+\cos(2x)}{2}\,dx = \int \frac{1}{2}\,dx + \int \cos(2x)\,dx =$$

$\boxed{\text{Merke: } \cos^2(x) = \frac{1+\cos(2x)}{2}}$

$$\cancel{\cos^2(x) = 1 - \sin^2(x) = 1 - \sin(x)\sin(x) = 1} = \underline{\underline{\frac{1}{2}x + \frac{1}{2}\sin(2x) + C}}$$

A8.4/13 PBZ

$$\int_4^8 \frac{y\,dy}{y^2-2y-3}$$

DAZU PBZ: $\frac{y}{y^2-2y-3} = \frac{f(y)}{g(y)}$

1) Zählergrad < Nennergrad? Ja ✓

2) Linearfaktoren: Nullstellen des Nenners finden: $y^2-2y-3 = 0$ $\Rightarrow y_1 = -1$, $\Rightarrow y_2 = 3$ evtl. Mitternachtsformel $\boxed{\text{Einfache Nullstellen}}$

$$\Rightarrow y^2-2y-3 = (y-(-1))(y-3)$$

4) $\frac{f(y)}{g(y)} = \frac{y}{y^2-2y-3} = \frac{y}{(y+1)(y-3)} = \frac{A}{(y+1)} + \frac{B}{(y-3)} \overset{5)}{=} \frac{A(y-3)}{(y+1)(y-3)} + \frac{B(y+1)}{(y+1)(y-3)}$

Koeffizienten einführen und gleichsetzen

3) Den Linf. Partialbrüche zuordnen

$(y+1) \longrightarrow \frac{A}{}$ $(y-3) \longrightarrow \frac{B}{}$

5) • (Hauptnenner)

~~4) Koeffizientenvergleich~~ $\Rightarrow y = A(y-3) + B(y+1) = Ay - 3A + By + B$

6) Koeffizientenvgl. $\underbrace{0}+\underbrace{1}\cdot y = y\underbrace{(A+B)}\underbrace{-3A+B}$

$$\Rightarrow \left\| \begin{array}{l} 0 = -3A + B \\ 1 = A + B \end{array} \right. \quad \Rightarrow 0 = -3(1-B) + B \Rightarrow -3 + 3B + B \overset{!}{=} 0$$

$\Rightarrow A = 1 - B$ $\qquad 4B = 3$, $B = 3/4$

$\Rightarrow A = 1 - 3/4 = 1/4$

7) $\Rightarrow \int \frac{y}{y^2-2y-3}\,dy$ lässt sich schreiben als

$$\int \left(\frac{1/4}{(y+1)} + \frac{3/4}{(y-3)}\right) dy$$

$$\Rightarrow \int_4^8 \left(\frac{1}{4(y+1)} + \frac{3}{4(y-3)}\right) dy = \frac{1}{4}\int_4^8 \frac{1}{(y+1)}\,dy + \frac{3}{4}\int_4^8 \frac{1}{y-3}\,dy = \frac{1}{4}\ln|y+1| + \frac{3}{4}\ln|y-3|$$

$$= \left[\frac{1}{4}\ln|y+1|\right]_{y=4}^{y=8} + \left[\frac{3}{4}\ln|y-3|\right]_{y=4}^{y=8} = \frac{1}{4}\left(\ln(9) - \ln(5)\right) + \frac{3}{4}\left(\ln(5) - \underbrace{\ln(1)}_{=0}\right)$$

$$\boxed{\begin{array}{l} a\ln(b) = \ln(b^a) \\ \ln(a) + \ln(b) = \ln(a\cdot b) \\ \ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right) \end{array}}$$

$$= \frac{1}{4}\left(\ln(9) - \ln(5) + 3\ln(5)\right) =$$

$$= \frac{1}{4}\left(\ln\left(\frac{9}{5}\cdot 5^3\right)\right) = \frac{1}{4}\ln\left(\frac{9}{25}\right) \quad \frac{1}{4}\ln(9\cdot 25) = \frac{1}{4}(3^2\cdot 5^2)$$

$$= \frac{1}{4}\ln(15^2) = \underline{\underline{\frac{\ln(15)}{2}}}$$

A8.7/4 SUB $\qquad$ Problem: $x \to 4 \Rightarrow \text{"}\frac{1}{0}\text{"}$

$$\int_0^4 \frac{dx}{\sqrt{4-x}} = \frac{\frac{1}{(1/2-1)}}{(4-x)^{(\frac{1}{2}-1)}} = \frac{2}{(4-x)^{-1/2}} = 2\sqrt{4-x} \quad \left[-2\sqrt{4-x}\right]_{x=0}^{x=4} =$$

$$= -2\left[0 - \sqrt{4}\right] = \underline{\underline{+4}}$$

$$\int_0^4 (4-x)^{-\frac{1}{2}}\,dx = \int_{u(0)}^{u(4)} u^{-1/2}(-1)\,du = \int_4^0 -u^{-1/2}\,du = \left[-2u^{+1/2}\right]_{u=4}^{u=0} = 0 - (-2\cdot\sqrt{4}) = \underline{\underline{4}}$$

$u(x) = 4 - x \Rightarrow \frac{du}{dx} = -1 \Rightarrow dx = -1\cdot du$

STOP

$u(4) = 4 - 4 = 0$
$u(0) = 4 - 0 = 4$

8.7/10

$$\int_{-\infty}^2 \frac{2}{x^2+4}\,dx = \lim_{a\to-\infty}\int_a^2 \frac{2}{(x+2)(x-2)}\,dx \overset{PBZ}{=} \lim_{a\to-\infty}\int_a^2 \left(\frac{(-1/2)}{(x+2)} + \frac{(1/2)}{(x-2)}\right) dx =$$

PBZ:

$$\frac{2}{(x+2)(x-2)} = \frac{A}{(x+2)} + \frac{B}{(x-2)} = \frac{A(x-2)}{(x+2)(x-2)} + \frac{B(x+2)}{(x+2)(x-2)}$$

$$\Rightarrow 2 = A(x-2) + B(x+2)$$

$$2 = Ax - 2A + Bx + 2B$$

8.7|10 $\quad \frac{d}{dx}\left(\arctan(x)\right) = \frac{1}{x^2+1}$

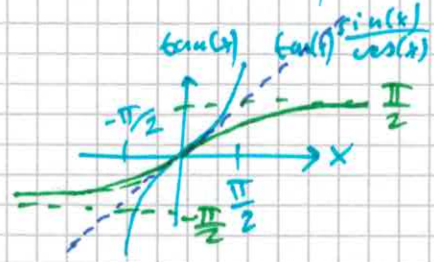
$$\int \frac{2}{x^2+4}\,dx = \int \frac{2/4}{\frac{x^2+4}{4}}\,dx = \int \frac{1/2}{\left(\frac{x}{2}\right)^2+1}\,dx = \arctan\left(\frac{x}{2}\right) + C$$

$$\Rightarrow \int_{-\infty}^{2} \frac{2}{x^2+4}\,dx = \lim_{a\to-\infty}\left[\arctan\left(\frac{x}{2}\right)\right]_{x=a}^{x=2} = \lim_{a\to-\infty}\left(\arctan(1) - \arctan\left(\frac{a}{2}\right)\right) =$$

$\frac{\pi}{4} = \arctan(1)$: $\quad \tan(x) = 1 \Rightarrow \sin(x) = \cos(x) \Rightarrow x = \frac{\pi}{4}$

$$\tan\left(\frac{\pi}{4}\right) = 1$$

$$\frac{\pi}{4} = \arctan\left(\tan\left(\frac{\pi}{4}\right)\right) = \arctan(1)$$



$$= \frac{\pi}{4} - \lim_{a\to-\infty}\left(\arctan\left(\frac{a}{2}\right)\right) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) =$$

$$= \underline{\underline{\frac{3\pi}{4}}}$$