Zusätzliche Aufgabe 4

(1) a) Wähle $f(x) = \ln x$ und $g'(x) = x$

$$\int x \cdot \ln x \, dx = \frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx = \underline{\underline{\frac{1}{2}x^2\left(\ln x - \frac{1}{2}\right) + C}}$$

b) Wähle $f(x) = \ln x$ und $g'(x) = x \cdot \ln x$

$$\int (\overset{\to a)}{x \ln x}) \cdot \ln x \, dx = \left(\frac{1}{2}x^2\left(\ln x - \frac{1}{2}\right)\right) \cdot \ln x - \int \left(\frac{1}{2}x^2\left(\ln x - \frac{1}{2}\right)\right) \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{2}x^2 \ln x\left(\ln x - \frac{1}{2}\right) - \int \frac{1}{2}\overset{\to a)}{x \cdot \ln x} - \frac{1}{4}x \, dx$$

$$= \frac{1}{2}x^2 \ln x\left(\ln x - \frac{1}{2}\right) - \frac{1}{4}x^2\left(\ln x - \frac{1}{2}\right) + \frac{1}{8}x^2 + C$$

$$= \frac{1}{2}x^2\left((\ln x)^2 - \frac{1}{2}\ln x - \frac{1}{2}\ln x + \frac{1}{4} + \frac{1}{4}\right) + C$$

$$= \underline{\underline{\frac{1}{2}x^2\left((\ln x)^2 - \ln x + \frac{1}{2}\right) + C}}$$

c) Wähle $f(x) = \ln x$ und $g'(x) = \frac{1}{x^2}$

$$\int \ln x \cdot \frac{1}{x^2}\, dx = -\frac{1}{x} \cdot \ln x - \int -\frac{1}{x} \cdot \frac{1}{x}\, dx = \underline{\underline{-\frac{1}{x}(\ln x + 1) + C}}$$

d) $$\int \frac{1}{x^2} \cdot \ln x \, dx = \frac{\ln x}{x} - \frac{1}{x} + 2\int \frac{\ln x}{x^2}\, dx + \frac{2}{x} \quad \Big| -\int \frac{\ln x}{x^2}\, dx \quad \Big| -\frac{\ln x}{x} - \frac{1}{x}$$

$$-\frac{\ln x}{x} - \frac{1}{x} = \int \frac{\ln x}{x^2}\, dx$$

$$\to \quad \underline{\underline{\int \frac{\ln x}{x^2}\, dx = -\frac{1}{x}(\ln x + 1)}} + C$$

ML Z4 A2

2a) 1) Bestimmung der Koeffizienten A & B

$$f(x) = \frac{5x-13}{(x-3)(x-2)} = \frac{A}{(x-3)} + \frac{B}{(x-2)} \quad |\text{Bruchenweiterung}$$

$$5x - 13 = A(x-2) + B(x-3) = Ax - 2A + Bx - 3B$$

↳ Koeffizientenvergleich: I – $5 = A + B \rightarrow A = 5 - B$

II – $-13 = -2A - 3B \rightarrow 13 = 2A + 3B$

I in II $13 = 10 - 2B + 3B$

$\underline{B = 3} \rightarrow \underline{A = 2}$

2) Integrieren

$$\int \frac{5x-13}{(x-3)(x-2)}\,dx = \int \frac{2}{(x-3)}\,dx + \int \frac{3}{(x-2)}\,dx$$

$$= \underline{\underline{2 \cdot \ln|x-3| + 3 \cdot \ln|x-2| + C}}$$

2b) 1) Nenner ausklammern: $x^2 - 3x + 2 = (x-2)\cdot(x-1)$

2) Bestimmung der Koeffizienten A & B

$$f(x) = \frac{5x-7}{x^2-3x+2} = \frac{A}{(x-2)} + \frac{B}{(x-1)} \quad |\text{Bruchenweiterung}$$

$$5x - 7 = A(x-1) + B(x-2)$$

↳ Koeffvgl: I $5 = A + B$

II $-7 = -A - 2B \rightarrow A = 7 - 2B$

II in I $5 = 7 - 2B + B \rightarrow \underline{B = 2} \rightarrow \underline{A = 3}$

3) Integrieren

$$\int \frac{5x-7}{x^2-3x+2}\,dx = \int \frac{3}{(x-2)}\,dx + \int \frac{2}{(x-1)}\,dx$$

$$= \underline{\underline{3\ln|x-2| + 2\ln|x-1| + C}}$$

2c) 1) Polynomdivision → Zähler < Nenner

$$(2x^2 - x - 3) : (x^2 - 3x + 2) = 2 + \frac{5x-7}{x^2-3x+2}$$
$$-(2x^2 - 6x + 4)$$
$$5x - 7$$

2) Integrieren

$$\int \frac{2x^2-x-3}{x^2-3x+2}\,dx = \int 2\,dx + \underbrace{\int \frac{5x-7}{x^2-3x+2}\,dx}_{\text{Herleitung Aufg 2b)}}$$

$$= \underline{\underline{2x + 3\ln|x-2| + 2\ln|x-1| + C}}$$

2d) 1) Polynomdivision → Zähler < Nenner

$$\begin{array}{l} 3x^2 : (x^2 - 3x + 2) = 3 + \frac{9x - 6}{(x^2 - 3x + 2)} \\ \underline{-(3x^2 - 9x + 6)} \\ \quad 9x - 6 \end{array}$$

2) Nenner ausklammern: $x^2 - 3x + 2 = (x-2)(x-1)$

3) Bestimmung der Koeff A & B

$$g(x) = \frac{9x+6}{x^2-3x+2} = \frac{A}{(x-2)} + \frac{B}{(x-1)} \quad | \text{ Bruchenweiterung}$$

$$9x - 6 = A(x-1) + B(x-2)$$

↳ Koeffvgl: $\text{I} \quad 9 = A + B$

$\text{II} \quad -6 = -A - 2B \rightarrow A = 6 - 2B$

$\text{II in I} \quad 9 = 6 - 2B + B \rightarrow \underline{B = -3} \rightarrow \underline{A = 12}$

4) Integrieren

$$\int \frac{3x^2}{x^2-3x+2}\,dx = \int 3\,dx + \int \frac{9x-6}{x^2-3x+2}\,dx$$

$$= \int 3\,dx + \int \frac{12}{x-2}\,dx - \int \frac{3}{x-1}\,dx$$

$$= \underline{\underline{3x + 12\ln|x-2| - 3\ln|x-1| + C}}$$

2e) 1) Bestimmung der Koeff A & B & C

$$f(x) = \frac{2}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-2)} \quad | \text{ Bruchenweiterung}$$

$$2 = A(x-1)(x-2) + B\cdot x\cdot(x-2) + C\cdot x\cdot(x-1)$$

$$\underline{2} = \underline{Ax^2} \underline{- 3Ax} + \underline{2A} + \underline{Bx^2} \underline{- 2Bx} + \underline{Cx^2} \underline{- Cx}$$

↳ Koeffvgl: $\text{I} \quad 2 = 2A \rightarrow \underline{A = 1}$

$\text{II} \quad 0 = -3A - 2B - C = -3 - 2B - C \rightarrow C = -3 - 2B$

$\text{III} \quad 0 = A + B + C = 1 + B + C$

$\text{II in III} \quad 0 = 1 + B - 3 - 2B \rightarrow \underline{B = -2} \rightarrow \underline{C = 1}$

2) Integrieren

$$\int \frac{2}{x(x-1)(x-2)}\,dx = \int \frac{1}{x}\,dx - \int \frac{2}{(x-1)}\,dx + \int \frac{1}{(x-2)}\,dx$$

$$= \underline{\underline{\ln|x| - 2\ln|x-1| + \ln|x-2| + C}}$$

2f) 1) Bestimmung der Koeff A & B

$$f(x) = \frac{x}{x^3 - 3x^2 + 2x} = \frac{\cancel{x}}{\cancel{x}(x^2 - 3x + 2)}$$

$$= \frac{1}{x^2 - 3x^2 + 2} = \frac{A}{(x-2)} + \frac{B}{(x-1)} \quad | \text{ Brucherweiterung}$$

$$1 = A(x-1) + B(x-2)$$

↳ Koeffvgl: I $\quad 0 = A + B \quad \rightarrow -A = B$

II $\quad 1 = -A - 2B$

I in II $\quad 1 = B - 2B \quad \rightarrow \underline{B = -1} \quad \rightarrow \underline{A = 1}$

2) Integrieren

$$\int \frac{x}{x^3 - 3x^2 + 2x}\,dx = \int \frac{1}{x-2}\,dx - \int \frac{1}{x-1}\,dx$$

$$= \underline{\underline{\ln|x-2| - \ln|x-1| + C}}$$

2g) 1) Bestimmung der Koeff A & B

$$f(x) = \frac{x}{(x-2)^2} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} \quad | \text{ Brucherweiterung}$$

$$x = A(x-2) + B$$

↳ Koeffvgl I $\quad 1 = A$

II $\quad 0 = -2A + B = -2 + B \rightarrow \underline{B = 2}$

2) Integrieren

$$\int \frac{x}{(x-2)^2}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{2}{(x-2)^2}\,dx$$

$$= \underline{\underline{\ln|x-2| - \frac{2}{(x-2)} + C}}$$

③ a) $F(x) = \int f(x)\,dx = \frac{1}{-p+1} \cdot x^{-p+1} + C$ für $p \neq 1$ und $p > 0$

$$\int_{\varepsilon}^{1} \frac{1}{x^p}\,dx = \left[\frac{1}{-p+1} \cdot x^{-p+1}\right]_{\varepsilon}^{1} = \underline{\underline{\frac{1}{-p+1}\left(1 - \varepsilon^{-p+1}\right)}}$$

für $p=1$: $\int_{\varepsilon}^{1} \frac{1}{x}\,dx = \left[\ln x\right]_{\varepsilon}^{1} = \underline{\underline{-\ln \varepsilon}}$

b) $\lim\limits_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{1}{x^p}\,dx = \lim\limits_{\varepsilon \to 0^+} \frac{1}{-p+1}\left(1 - \varepsilon^{-p+1}\right)$

$-p+1 > 0$: $= \frac{1}{-p+1}$

$-p+1 < 0$: $= +\infty$

d.h. das uneigentliche Integral konvergiert für $-p+1 > 0 \to \underline{\underline{0 < p < 1}}$

für $p=1$: $\lim\limits_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{1}{x}\,dx = \lim\limits_{\varepsilon \to 0^+} \left[\ln x\right]_{\varepsilon}^{1} = \underline{\underline{+\infty}}$

d.h. für $p=1$ strebt das Integral gegen $+\infty$