ETH Zürich HS 2015
D-MATH
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Mathematical Finance
Exercise Sheet 6

Please hand in until Friday, 4.12.2015, 12:00.

Consider a finite horizon model $T < \infty$ with one risky and one risk-free asset. The price of the risky asset S is governed by a geometric Brownian motion with volatility σ and drift μ. The bond price B is supposed to be normalized: $B \equiv 1$. Take a strictly increasing and strictly concave utility function U on $\mathbb{R}$. Given an initial wealth $x \in \mathbb{R}$, the utility maximization problem is

$$\sup_{\varphi \in \mathcal{A}} E\left[U(x + \int_0^T \varphi_u dS_u)\right]. \tag{1}$$

Exercise 6-1

The value function of the utility maximization problem is defined by

$$u(t,x) = \operatorname{ess\,sup}_{\varphi \in \mathcal{A}} E\left[U(X_T^{x,\varphi}) \mid \mathcal{F}_t\right], \quad (t,x) \in [0,T] \times \operatorname{dom}(U), \tag{2}$$

where $X_T^{x,\varphi} := x + \int_t^T \varphi_u dS_u$. Show that the martingale optimality principle follows from the dynamic programming principle. You may assume that, for every $(t,x) \in [0,T] \times \operatorname{dom}(U)$, the (essential) supremum in (2) is attained for some $\varphi \in \mathcal{A}$.

Exercise 6-2

Assume that the utility function is of the form

$$U(x) = -e^{-\alpha x}, \quad x \in \mathbb{R},\ \alpha > 0.$$

Solve the utility maximization (1) via HJB.

Exercise 6-3

Let Q be the unique equivalent martingale measure. Show that if there exists a maximizer $\widehat{X}_T = x + \int_0^T \widehat{\varphi}_u dS_u$ for (1), then

$$\frac{dQ}{dP} = \frac{1}{c} U'(\widehat{X}_T)$$

for some $c > 0$. Determine the constant c for $U(x) = -e^{-\alpha x}$, $x \in \mathbb{R}$, $\alpha > 0$, and a replicating portfolio for $\widehat{X}_T$.

Exercise 6-4

Solve the utility maximization problem (1) for a logaritmic utility: $U(x) = \log(x)$, $x \in \mathbb{R}_+$.

Exercise 6-5

Given two points x_1 and x_2 in $\mathbb{R}^d$, the shortest path connecting x_1 to x_2 is a straight line. Verify this by formulating the statement as an optimal control problem and solving it.