ETH Zürich HS 2015
D-MATH
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Mathematical Finance
Solutions Sheet 3

Solution 3-1

a) The market price of risk equation is

$$\sigma_t \lambda_t = \mu_t - r_t \mathbf{1} =: b_t, \ t \in [0, T],$$

where $\mathbf{1}$ denotes the vector whose entries are all equal to 1. By the assumption, this equation has a P-almost surely unique solution $\lambda_t = \sigma_t^{-1} b_t$ for every $t \in [0, T]$.

b) The equivalent local martingale measures Q are parametrized via

$$Z_t^Q = \frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \mathcal{E}\left(- \int b^T (\sigma \sigma^T)^{-1} \sigma dW + \int \nu dW \right)_t, \ t \in [0, T],$$

with $\sigma \nu = 0$. Since σ is invertible, $\nu = 0$ and

$$Z^Q = \mathcal{E}\left(- \int b^T (\sigma^T)^{-1} dW \right) = \mathcal{E}\left(- \int \lambda^T dW \right)$$

is unique. So, Q is unique.

c) Let $H \in L^\infty(\mathcal{F}_T)$ denote the discounted payoff at time T and define the Q-martingale

$$Y_t := E_Q[H | \mathcal{F}_t], \quad 0 \le t \le T.$$

Denoting $Z_t^Q = \frac{dQ}{dP}\big|_{\mathcal{F}_t} = \mathcal{E}\left(-\int \lambda^T dW\right)_t$ (by b) and applying Bayes' rule we deduce that $Y_t Z_t^Q$ is a P-martingale. By the standard representation theorem for martingales, we can therefore write $Y Z^Q$ as a stochastic integral with respect to W, that is

$$Y_t Z_t^Q = Y_0 + \int_0^t \psi_s dW_s.$$

By Ito's formula we obtain

$$dY_t = \left(\frac{1}{Z_t^Q} \psi_t + Y_t \lambda_t^T \right) dW^Q,$$

where W^Q denotes the Girsanov transformed Q-Brownian motion. Observe that this is a martingale representation with respect to W^Q. In order to show attainability of H we have to find some admissible θ such that

$$dY_t = \sum_{i=1}^d \theta_t^i dS_t^i$$

is satisfied. Since $dS_t^i = S_t^i \sum_{j=1}^d \sigma_t^{ij} d(W_t^Q)^j$ and since σ is invertible, we can define θ^i by

$$\theta^i = \frac{\left(\left(\frac{1}{Z^Q}\psi + Y\lambda^T\right)\sigma^{-1}\right)_i}{S^i}$$

yielding

$$Y_t = E_Q[H] + \int_0^t \sum_{i=1}^d \theta_t^i dS_t^i.$$

Note that admissibility is satisfied since the left hand side of the above equation is a.s. bounded, implying that the gains process $\int_0^t \sum_{i=1}^d \theta_t^i dS_t^i$ is a.s. bounded (from below).

Solution 3-2

There exists a measure Q such that the *discounted* stock price process, $S = (S_t)_{t\in[0,T]}$,

$$S_t = S_0 \exp(\sigma W_t^Q - \frac{1}{2}\sigma^2 t),$$

is a martingale. W^Q denotes a Q-Brownian motion. The *undiscounted* stock price process $\widetilde{S}$ is given by

$$\widetilde{S}_t = e^{rt} S_t = e^{rt} S_0 \exp(\sigma W_t^Q - \frac{1}{2}\sigma^2 t), \ t \in [0, T].$$

We have

$$\widetilde{S}_T = e^{r(T-t)} \widetilde{S}_t \exp\left(\sigma(W_T^Q - W_t^Q) - \frac{1}{2}\sigma^2(T-t)\right), \ t \in [0, T].$$

The value of a power option, payoff $h(\widetilde{S}_T) = \widetilde{S}_T^p$, at time t is its discounted expected value:

$$\begin{aligned} V_t &= e^{-r(T-t)} E_Q[h(\widetilde{S}_T)|\mathcal{F}_t] \\ &= e^{-r(T-t)} E_Q[\widetilde{S}_T^p|\mathcal{F}_t] \\ &= e^{-r(T-t)} \ \widetilde{S}_t^p \ e^{pr(T-t)} \ e^{-\frac{1}{2}p\sigma^2(T-t)} \ E_Q[e^{p\sigma(W_T^Q - W_t^Q)} \mid \mathcal{F}_t] \\ &= e^{-r(T-t)} \ \widetilde{S}_t^p \ e^{pr(T-t)} \ e^{-\frac{1}{2}p\sigma^2(T-t)} \ e^{\frac{1}{2}p^2\sigma^2(T-t)} \\ &= \widetilde{S}_t^p \ e^{(r+\frac{1}{2}\sigma^2 p)(p-1)(T-t)}. \end{aligned}$$

The Δ-hedging strategy $\varphi = (\theta, \eta)$,

$$dV_t := \theta_t d\widetilde{S}_t + (V_t - \theta_t \widetilde{S}_t) r dt := \theta_t d\widetilde{S}_t + \eta_t r dt,$$

is

$$\begin{aligned} \theta_t &= \frac{\partial V_t}{\partial \widetilde{S}_t} = p\widetilde{S}_t^{p-1} e^{(r+\frac{1}{2}\sigma^2 p)(p-1)(T-t)}, \\ \eta_t &= e^{-rt}(V_t - \theta_t \widetilde{S}_t) = e^{-rt}(1-p)\widetilde{S}_t^p e^{(r+\frac{1}{2}\sigma^2 p)(p-1)(T-t)}, \end{aligned}$$

where $t \in [0, T]$.

Solution 3-3

Under the assumptions, V is a supermartingale with $\sup_{t\in[0,T]}|V_t| \in L^1$, and so has a Doob-Meyer decomposition

$$V_t = V_0 + \widetilde{M}_t - \widetilde{A}_t,$$

where $\widetilde{M}$ is a martingale vanishing at zero, and $\widetilde{A}$ an integrable predictable increasing process, also vanishing at zero. We have

$$\sup_{0\le t\le T} |\widetilde{M}_t| \le \sup_{0\le t\le T} E[\sup_{0\le s\le T} |U_s| \mid \mathcal{F}_t] + |V_0| + \widetilde{A}_T,$$

so, $\widetilde{M} \in H_0^1$. And since $U_t \le V_t = V_0 + \widetilde{M}_t - \widetilde{A}_t$,

$$\begin{aligned}
\inf_{M\in H_0^1} E[\sup_{0\le t\le T}(U_t - M_t)] &\le E[\sup_{0\le t\le T}(U_t - \widetilde{M}_t)] \\
&\le E[\sup_{0\le t\le T}(V_t - \widetilde{M}_t)] \\
&\le E[\sup_{0\le t\le T}(V_0 - \widetilde{A}_t)] \\
&= V_0.
\end{aligned}$$

On the other, for any other $M \in H_0^1$, we have

$$V_0 = \sup_{0\le\tau\le T} EU_\tau = \sup_{0\le\tau\le T} E[U_\tau - M_\tau] \le E[\sup_{0\le t\le T}(U_t - M_t)],$$

i.e.,

$$V_0 \le \inf_{M\in H_0^1} E[\sup_{0\le t\le T}(U_t - M_t)].$$

Solution 3-4

Assume a density f for S_T. Recall Leibniz integration rule

$$\frac{\partial}{\partial K}\int_{a(K)}^{b(K)} f(x,K)dx = \frac{db(K)}{dK} f(b(K),K) - \frac{da(K)}{dK} f(a(K),K) + \int_{a(K)}^{b(K)} \frac{\partial}{\partial K} f(x,K)dx.$$

For a call option

$$C(K) = \int_0^\infty (x-K)^+ f(x)dx = \int_K^\infty (x-K)f(x)dx$$

we get

$$\frac{\partial C}{\partial K} = 0 - (K-K)f(K) - \int_K^\infty f(x)dx = -\int_K^\infty f(x)dx.$$

Since

$$1 = \int_0^\infty f(x)dx = \int_0^K f(x)dx + \int_K^\infty f(x)dx,$$

by Fundamental Theorem of Calculus, we get "Breeden-Litzenberger formula"

$$\frac{\partial^2 C}{\partial K^2}(K) = \frac{\partial}{\partial K}[\int_0^K f(x)dx - 1] = f(K).$$

Similarly for a put option

$$\frac{\partial^2 P}{\partial K^2}(K) = f(K).$$

So,

$$E[w(S_T)] = \int_0^\infty w(K)f(K)dK = \int_0^{S_0} w(K)\frac{\partial^2 P}{\partial K^2}(K)dK + \int_{S_0}^\infty w(K)\frac{\partial^2 C}{\partial K^2}(K)dK = \cdots$$

and, by integration by parts,

$$\begin{aligned}
\cdots &= [w(K)\int_0^K f(x)dx]\Big|_0^{S_0} - \int_0^{S_0} w'(K)\frac{\partial P}{\partial K}(K)dK + [w(K)(\int_0^K f(x)dx - 1)]\Big|_{S_0}^\infty - \int_{S_0}^\infty w'(K)\frac{\partial C}{\partial K}(K)dK \\
&= w(S_0) - \int_0^{S_0} w'(K)\frac{\partial P}{\partial K}(K)dK - \int_{S_0}^\infty w'(K)\frac{\partial C}{\partial K}(K)dK \\
&= w(S_0) - [w'(K)P(K)]\Big|_0^{S_0} + \int_0^{S_0} w''(K)P(K)dK - [w'(K)C(K)]\Big|_{S_0}^\infty + \int_{S_0}^\infty w''(K)C(K)dK \\
&= w(S_0) + \int_0^{S_0} w''(K)P(K)dK + \int_{S_0}^\infty w''(K)C(K)dK.
\end{aligned}$$

Solution 3-5

We may assume $(x_n) \subset \text{int dom}(f)$. We have

$$f_n(y) \geq f_n(x_n) + \langle x_n^*, y - x_n \rangle \ \forall y \in X \ \forall n \in \mathbb{N}. \tag{1}$$

There exists $\delta > 0$ such that $f_n \to f$ uniformly on $B(x, \delta) \subset \text{int dom}(f)$. Choose $y_n = x_n + \frac{\delta}{2}\frac{x_n^*}{||x_n^*||}1_{\{x_n^* \neq 0\}}$. Then

$$f_n(x_n + \frac{\delta}{2}\frac{x_n^*}{||x_n^*||}1_{\{x_n^* \neq 0\}}) - f_n(x_n) \geq \frac{\delta}{2}||x_n^*||1_{\{x_n^* \neq 0\}}.$$

Taking limits on both sides yields

$$f(x + \frac{\delta}{2}\frac{x^*}{||x^*||}1_{\{x^* \neq 0\}}) - f(x) \geq \frac{\delta}{2}\lim_{n\to\infty} ||x_n^*||1_{\{x_n^* \neq 0\}}$$

and $\lim_{n\to\infty} ||x_n^*|| = \infty$ would be a contradiction. So, (x_n^*) is bounded. Assume now that $x_n^* \to x^*$. Returning to inequality (1), it is sufficient to show that it holds for all $y \in B(x, \delta)$. We have

$$\begin{aligned}
f_n(y) &\geq f_n(x_n) + \langle x_n^*, y - x_n \rangle \\
&= f_n(x_n) + \langle x^*, y - x_n \rangle + \langle x_n^* - x^*, y - x_n \rangle,
\end{aligned}$$

where

$$\begin{aligned}
|\langle x_n^* - x^*, y - x_n \rangle| &\leq |\langle x_n^* - x^*, y \rangle| + |\langle x_n^* - x^*, x_n \rangle| \\
&\leq ||x_n^* - x^*||(\delta + \sup_n ||x_n||).
\end{aligned}$$

Letting $n \to \infty$ yields

$$f(y) \geq f(x) + \langle x^*, y - x \rangle \ \forall y \in B(x, \delta),$$

i.e., $x^* \in \partial f(x)$.