ETH Zürich HS 2015
D-MATH
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Mathematical Finance
Solutions Sheet 6

Solution 6-1

Consider any, potentially suboptimal, trading strategy $\varphi \in \mathcal{A}$, and fix $0 \le s \le t \le T$. As we are assuming that the supremum in

$$u(t, X_t^\varphi) = \operatorname{ess\,sup}_{\varphi \in \mathcal{A}} E\left[U(X_t^\varphi + \int_t^T \varphi_u dS_u) \mid \mathcal{F}_t\right]$$

is attained for some $\widehat{\psi} \in \mathcal{A}$ on $[t, T]$, we have

$$\begin{aligned} E[u(t, X_t^\varphi) \mid \mathcal{F}_s] &= E[E[U(X_t^\varphi + \int_t^T \widehat{\psi}_u dS_u) \mid \mathcal{F}_t] \mid \mathcal{F}_s] \\ &= E[U(X_s^\varphi + \int_s^t \varphi_u dS_u + \int_t^T \widehat{\psi}_u dS_u) \mid \mathcal{F}_s] \le u(s, X_s^\varphi). \end{aligned} \tag{1}$$

Suppose that the investor has followed the optimal trading strategy $\widehat{\varphi} \in \mathcal{A}$ on the sub-interval $[0, t]$. The dynamic programming principle states that if the investor is allowed to re-examine her portfolio choice taking account her present wealth $X_t^{\widehat{\varphi}}$, the optimal strategy that she will choose, $\widehat{\psi}$, coincides with her initial choice $\widehat{\varphi}$, which yields an equality in (1).

Solution 6-2

By, Ito's formula,

$$du(t, X_t^\varphi) = (u_t + u_x[\varphi_t S_t]\mu + \frac{1}{2}[\varphi_t S_t]^2)dt + u_x[\varphi_t S_t]\sigma dW_t,$$

for any $\varphi \in \mathcal{A}$. By the martingale optimality principle, the drift of value function (process) should be non-positive for any strategy and vanish for the optimal. The drift rate is a quadratic function of the risky position $\vartheta_t := \varphi_t S_t$. Taking pointwise maximum

$$\widehat{\vartheta}_t := \frac{\mu}{(-u_{xx}/u_x)\sigma}$$

and inserting $\widehat{\vartheta}$ back to the drift rate, which should vanish for the maximizing choice, leads us to the HJB equation:

$$u_t = \frac{u_x^2}{u_{xx}} \frac{\mu^2}{2\sigma^2}.$$

For the exponential utility we have

$$u(t, x) = \operatorname{ess\,sup}_\varphi E[-e^{-\alpha(x + \int_t^T \varphi_u dS_u)} \mid \mathcal{F}_t] = e^{-\alpha x}\phi(t).$$

So, the optimal risky position (resp. number of shares) is $\widehat{\vartheta} = \frac{\mu}{\alpha\sigma}$ (resp. $\widehat{\varphi}_t = \frac{\mu}{\alpha\sigma^2}\frac{1}{S_t}$) and HJB reduces to ODE

$$\phi' = \frac{\mu^2}{2\sigma^2}\phi.$$

The solution satisfying the terminal condition, $\phi(T) = -1$, is

$$\phi(t) = -\exp(-\frac{\mu^2}{2\sigma^2}(T-t)),$$

i.e.,

$$u(0,x) = -\exp(-\alpha x - \frac{\mu^2}{2\sigma^2}T).$$

Solution 6-3

Define the convex conjugate of U as

$$V(y) = \sup_{x\in\text{dom}(U)}\{U(x) - xy\},\ y > 0.$$

By the Fenchel inequality, $U(x) \le V(y) + xy$ for all $x \in \text{dom}(U)$ and $y > 0$. Hence,

$$E[U(X_T)] \le E[V(y\frac{dQ}{dP})] + E_Q[yX_T] \le E[V(y\frac{dQ}{dP})] + yx,$$

for every $x \in \text{dom}(U)$, $y > 0$ and wealth process X that is supermartingale under Q, and the equality is achieved if $y\frac{dQ}{dP} = U'(X_T)$ and $E_Q[X_T] = x$. Since U is strictly increasing and strictly concave, we have $c > 0$ for $c\frac{dQ}{dP} = U'(\widehat{X}_T)$.

Now, let $U(x) = -e^{-\alpha x}$. Then the inverse of marginal utility is $(U')^{-1}(x) = -\frac{1}{\alpha}\log(\frac{x}{\alpha})$ and we have

$$x = E_Q[\widehat{X}_T] = E[\frac{dQ}{dP}(U')^{-1}(c\frac{dQ}{dP})] = E[\frac{dQ}{dP}(-\frac{1}{\alpha}\log(\frac{c}{\alpha}\frac{dQ}{dP}))],$$

so,

$$c = \alpha e^{-\alpha x - E[\frac{dQ}{dP}\log(\frac{dQ}{dP})]}.$$

In Black-Scholes with zero interest rate, the equivalent martingale measure is given by

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \frac{\mu}{\sigma}dW_t - \int_0^T \frac{\mu^2}{2\sigma^2}dt\right). \tag{2}$$

We have

$$\begin{aligned}\widehat{X}_T = x + \int_0^T \varphi_t dS_t = -\frac{1}{\alpha}\log(\frac{c}{\alpha}\frac{dQ}{dP}) &= -\frac{\log(c/\alpha)}{\alpha} + \int_0^T \frac{\mu}{\alpha\sigma}dW_t + \int_0^T \frac{\mu^2}{2\alpha\sigma^2}dt\\ &= x + \int_0^T \frac{\mu}{\alpha\sigma^2}\frac{1}{S_t}(S_t\sigma dW_t + S_t\mu dt).\end{aligned}$$

So, $\widehat{\varphi}_t = \frac{\mu}{\alpha\sigma^2}\frac{1}{S_t}$.

Solution 6-4

We have $U(x) = \log(x)$, so the inverse of the marginal utility function $(U')^{-1}$ is $y \mapsto \frac{1}{y}$. Denote by Z the density process of Q w.r.t. P. The optimal wealth process is

$$X_t^{x,\varphi} = E_Q[\frac{1}{cZ_T}|\mathcal{F}_t] = E[\frac{Z_T}{Z_t}\frac{1}{cZ_T}|\mathcal{F}_t] = \frac{1}{cZ_t} =: M_t,\ 0 \le t \le T,$$

where c is s.t. $X_0^{x,\varphi} = x$, and so $c = 1/x$. Hence, $X_t^{x,\varphi} = x/Z_t$, $0 \le t \le T$. Now, from (2), we deduce by Ito formula that

$$dM_t = M_t \frac{\mu}{\sigma} dW_t.$$

As in the previous exercise, identifying this with the dynamics of the wealth process we get the optimal portfolio in terms of number of shares:

$$\varphi_t = \frac{\mu}{\sigma^2} \frac{X_t^{x,\varphi}}{S_t},\ 0 \le t \le T,$$

or equivalently as a proportion of the wealth:

$$\pi_t = \frac{\varphi_t S_t}{X_t^{x,\varphi}} = \frac{\mu}{\sigma^2},\ 0 \le\ t \le T.$$

Solution 6-5

We may think that we are moving a point that moves with a unit speed. Let $x = x(t)$ denote the position of point in $\mathbb{R}^d$. Then $x(0) = x_1$, $x(T) = x_2$, and we minimize

$$\int_0^T |\dot{x}(t)| dt = \int_0^T dt = T$$

for

$$\dot{x}(t) = \alpha(t),$$

where the control α takes values on the unit sphere $S^1 = \{(a_1, a_2) \in \mathbb{R}^2 : a_1^2 + a_2^2 = 1\}$. The Hamiltonian is $H(x(t), \lambda(t), \alpha(t)) := \alpha(t) \cdot \lambda(t) - 1$, where $\lambda(t)$ is the costate, and since

$$\dot{\lambda}(t) = -\frac{\partial}{\partial x} H(x(t), \lambda(t), \alpha(t)) = 0,$$

it is a constant. Since $x(t)$ is understood as the position of a point moving at a unit velocity, it is reasonable to assume that $\lambda^* \neq 0$ as it represents the momentum of the point in Hamiltonian mechanics. By the Pontryagin's maximum principle,

$$H(x^*(t), \lambda^*, \alpha^*(t)) = \max_{a \in S^1} H(x^*(t), \lambda^*, a) = \max_{a \in S^1} \{a \cdot \lambda^* - 1\},$$

which is maximized for $a^* = \frac{\lambda^*}{|\lambda^*|}$. Thus $\alpha^*(t)$ is equivalent to a constant a^*. We conclude that $x^*(t)$ is a line from x_1 to x_2.