ETH Zürich HS 2015
D-MATH
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Mathematical Finance
Solutions Sheet 7

Solution 7-1

a) Since the interest rate is zero, $\eta^0 = \varphi^0$. So, from the self-financing condition, we get

$$\eta_t^0 = -S_t d\varphi_t^b + (1-\lambda)S_t d\varphi_t^s, \; t \in [0,T].$$

Similarly, plugging in the dynamics of S, we get

$$d\eta_t = \mu\eta_t dt + \sigma\eta_t dW_t + S_t d\varphi^b - S_t d\varphi_t^s.$$

b) We assume that the value function depends on time t, the current value of safe position x and the current value of risky position y, $u = u(t,x,y)$. By, Ito's Formula,

$$\begin{aligned} du(t,\eta_t^0,\eta_t) &= u_t d_t + u_x d\eta_t^0 + u_y d\eta_t + \frac{1}{2}u_{yy} d\langle \eta,\eta\rangle \\ &= (u_t + \mu\eta_t u_y + \frac{1}{2}\sigma^2\eta^2 u_{yy})dt + S_t(u_y - u_x)d\varphi^b + S_t((1-\lambda)u_x - u_y)d\varphi_t^s + \sigma\eta_t u_y dW_t. \end{aligned}$$

By the martingale optimality principle, $u(t,\eta_t^0,\eta_t)$ must be a supermartingale for any $\eta \in \mathcal{A}$ and a martingale for the optimal $\widehat{\eta}$. So, it follows that $u_y - u_x \leq 0$ and $(1-\lambda)u_x - u_y \leq 0$, i.e.,

$$1 \leq \frac{u_x}{u_y} \leq \frac{1}{1-\lambda},$$

and for $\widehat{\eta}$, the drift must vanish in the interior of this region. We get

$$u_t + \mu\eta_t u_y + \frac{1}{2}\sigma^2\eta_t^2 u_{yy} = 0 \text{ on } 1 < \frac{u_x}{u_y} < \frac{1}{1-\lambda}.$$

Solution 7-2

As in the frictionless case (Exercise 6-2), by the scaling property of exponential utility, we may rewrite

$$u(t,x,y) = e^{-\alpha x} u(t,0,y).$$

In the frictionless case, the corresponding equivalent annuity is

$$\liminf_{T\to\infty} -\frac{1}{\alpha T}\log E[e^{-\alpha X_T^\phi}] = \frac{\mu^2}{2\alpha\sigma^2},$$

so, we expect similar behavior in the present setting:

$$u(t,\eta_t^0,\eta) = -e^{-\alpha\eta_t^0} e^{-\beta\eta_t^0}\phi(\eta_t). \tag{1}$$

Plugging (1) in HJB, we get

$$\frac{1}{2}\sigma^2 y^2 \phi''(y) + \mu y \phi'(y) \alpha\beta\phi(y) = 0 \text{ on } 1 < \frac{-\alpha\phi(y)}{\phi'(y)} < \frac{1}{1-\lambda}. \tag{2}$$

Denote $]l, m[:= \{y \in \mathbb{R} : 1 < \frac{-\alpha\phi(y)}{\phi'(y)} < \frac{1}{1-\lambda}\}$. We then have a free boundary problem:

$$\begin{aligned} &\frac{1}{2}\sigma^2 y^2 \phi''(y) + \mu y \phi'(y) \alpha\beta\phi(y) = 0 \text{ on } 1 < \frac{-\alpha\phi(y)}{\phi(y)} < \frac{1}{1-\lambda}, \\ &\phi'(l) + \alpha\phi'(l) = 0, \\ &\frac{1}{1-\lambda}\phi(m) + \alpha\phi'(m). \end{aligned}$$

The optimal boundaries are given by the smooth pasting condition:

$$\begin{aligned} &\phi''(l) + \alpha\phi'(l) = 0, \\ &\frac{1}{1-\lambda}\phi'(m) + \alpha\phi''(m). \end{aligned}$$

We get

$$-\frac{1}{2}\alpha\sigma^2\eta_{\alpha-}^2 + \mu\eta_{\alpha-}^2 - \beta = 0,$$

where $\eta_{\alpha-} := l$. Similar argument for m shows that the other solution of quadratic equation is $\eta_{\alpha+} := (1-\lambda)m$. Since they solve the same quadratic equation, they are related via

$$\mu_{\alpha\pm} = \frac{\mu}{\alpha\sigma} \pm \frac{1}{\alpha}\sqrt{\mu^2/\sigma^4 - 2\beta/(\alpha\sigma^2)}.$$

Solution 7-3

Plugging in our candidate solution, we get

$$\widetilde{S}_t = \frac{w(\log(\eta_t/\eta_{\alpha-}))}{\alpha\eta_t} S_t.$$

Let $Y_t := \log(\eta_t/\eta_{\alpha-})$. The process Y is a reflected Brownian motion on the interval $[0, \log(\frac{1}{1-\lambda}\frac{\eta_{\alpha+}}{\eta_{\alpha-}})]$. Indeed, in the interior of the interval, the dynamics Y coincide with those of Brownian motion, and since Y must stay in the interval, we have

$$dY_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t + dL_t - dU_t,$$

where L and U are non-decreasing local time processes, increasing only on $\{Y_t = 0\}$ and $\{Y_t = \log(\frac{1}{1-\lambda}\frac{\eta_{\alpha+}}{\eta_{\alpha-}})\}$ respectively. The initial state is, by the definition of no-trade region, $Y_0 = 0$ if $xS_0 \leq \eta_{\alpha-}$, $Y_0 = \log(\frac{1}{1-\lambda}\frac{\eta_{\alpha+}}{\eta_{\alpha-}})$ if $xS_0 \geq \frac{1}{1-\lambda}\eta_{\alpha+}$, and $Y_0 = \log(\frac{xS_0}{\eta_{\alpha-}})$ otherwise. Since $Y = \log(\eta/\eta_{\alpha-})$, we have $\widetilde{S} = \frac{w(Y)}{\alpha\eta_{\alpha-}e^Y}$, which fixes the initial value of $\widetilde{S}$. By Ito formula,

$$\frac{d(S_t/\alpha\eta_{\alpha-}e^Y)}{S_t/\alpha\eta_{\alpha-}e^Y} = -d(L_t - U_t)$$

and

$$\frac{dw(Y_t)}{w(Y_t)} = (\frac{w'(Y_t)}{w(Y_t)}(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}\sigma^2\frac{w''(Y_t)}{w(Y_t)})dt + \frac{w'(Y_t)}{w(Y_t)}\sigma dW_t + \frac{w'(Y_t)}{w(Y_t)}d(L_t - U_t).$$

Differentiating ODE for w ($w'' - w' = 2w'(w - \frac{\mu}{\sigma^2})$), the above expression reduces to

$$\frac{dw(Y_t)}{w(Y_t)} = \sigma^2 w'(\log(\eta_t/\eta_{\alpha-}))dt + \sigma \frac{w'(\eta_t/\eta_{\alpha-})}{w(\eta_t/\eta_{\alpha-})} dW_t + d(L_t - U_t)$$

and the assertion now follows by the integration by parts. Since $(w' - w)'$ is non-positive for $w \leq \frac{\mu}{\sigma^2}$ and positive for $w > \frac{\mu}{\sigma^2}$ and $w = w'$ on the boundaries, we have that the derivative of $w(y)/e^y$, that is $(w'(y) - w(y))/e^y$, is non-positive, so $w(y)/e^y$ is monotonic. Since $w(0) = \alpha\eta_{\alpha-}$ and $w(\log(\frac{1}{1-\lambda}\frac{\eta_{\alpha+}}{\eta_{\alpha-}})) = \alpha\eta_{\alpha+}$, the process $\widetilde{S} = \frac{w(Y)}{\alpha\eta_{\alpha-}e^Y}$ stays in the bid-ask spread $[(1-\lambda)S, S]$.

Solution 7-4

The density of an equivalent local martingale measure $\widetilde{Q}$ for $\widetilde{S}$ is

$$Z_T = \exp(-\int_0^T \sigma w dW_t - \frac{1}{2}\int_0^T \sigma^2 w^2 dt).$$

Since $\sigma\frac{w'(\eta_t/\eta_{\alpha-})}{w(\eta_t/\eta_{\alpha-})}$ is uniformly bounded and $(1-S) \leq \widetilde{S} \leq S$, the local $\widetilde{Q}$-martingale $\widetilde{X}_t^\varphi = \widetilde{X}_0^\varphi + \int_0^t \varphi_t d\widetilde{S}_t$, is a true martingale for every admissible φ. As in the frictionless case (Exercise 6-3), by the Jensen's inequality and martingale property, we have

$$E[e^{-\alpha\widetilde{X}_T^\varphi}] = E_{\widetilde{Q}}[e^{-\alpha\widetilde{X}_T^\varphi - \log(Z_T)}] \geq e^{-\alpha E_{\widetilde{Q}}[\widetilde{X}_T^\varphi] - E_{\widetilde{Q}}[\log(Z_T)]} \geq e^{-\alpha\widetilde{X}_0^\varphi - E_{\widetilde{Q}}[\log(Z_T)]},$$

which yields an upper bound for equivalent annuities

$$\liminf_{T\to\infty} \frac{-1}{\alpha T}\log(E[e^{-\alpha\widetilde{X}_T^\varphi}]) \leq \liminf_{T\to\infty}\frac{1}{T}(\widetilde{X}_0^\varphi + \frac{1}{\alpha}E_{\widetilde{Q}}[\log(Z_T)]). \tag{3}$$

On the other hand, for $\widehat{\eta}$, and respective wealth process,

$$\widehat{X}_t = (x + x\widetilde{S}_0) + \int_0^t \widehat{\eta}\sigma^2 w'(\log(\eta_t/\eta_{\alpha-}))dt + \int_0^t \widehat{\eta}\sigma\frac{w'(\eta_t/\eta_{\alpha-})}{w(\eta_t/\eta_{\alpha-})}dW_t,$$

we have

$$\begin{aligned} e^{-\alpha\widehat{X}_T} &= e^{-\alpha\widehat{X}_0}\exp(-\int_0^T \widehat{\eta}\sigma^2 w'(\log(\eta_t/\eta_{\alpha-}))dt + \int_0^T \widehat{\eta}\sigma\frac{w'(\eta_t/\eta_{\alpha-})}{w(\eta_t/\eta_{\alpha-})}dW_t) \\ &= e^{-\alpha\widehat{X}_0}\exp(-\int_0^T \sigma^2 ww' dW_t - \frac{1}{2}\int_0^T \sigma w' dt) = \cdots \end{aligned} \tag{4}$$

by the dynamics of $w(\log(\eta/\eta_{\alpha-}))$, we have

$$\begin{aligned} \int_{\log(\eta_0/\eta_{\alpha-})}^{\log(\eta_T/\eta_{\alpha-})} (w(z) - w'(z))dz &= \int_0^T \big((\mu - \frac{1}{2}\sigma^2) + (w - w')\frac{1}{2}\sigma^2(w' - w'')\big)dt + \int_0^T \sigma(w - w')dW_t \\ &= \int_0^T \big((\mu - \frac{1}{2}\sigma^2)w + \frac{1}{2}\sigma^2 w' - \sigma^2 ww'\big)dt + \int_0^T \sigma(w - w')dW_t \end{aligned}$$

so, (4) is equal to

$$\begin{aligned} \cdots &= e^{-\alpha\widehat{X}_0}\exp(\int_{\log(\eta_0/\eta_{\alpha-})}^{\log(\eta_T/\eta_{\alpha-})} (w(z) - w'(z))dz + \frac{1}{2}\sigma^2\int_0^T (-w' - (2\frac{\mu}{\sigma^2} - 1)w)dt - \int_0^T \sigma w dW_t) \\ &= e^{-\alpha\widehat{X}_0}e^{-\alpha\beta T}Z_T\exp(\int_{\log(\eta_0/\eta_{\alpha-})}^{\log(\eta_T/\eta_{\alpha-})} (w(z) - w'(z))dz) \end{aligned}$$

Taking expectations, we get

$$E[e^{-\alpha\widehat{X}_T}] = e^{-\alpha\widehat{X}_0}e^{-\alpha\beta T}E_{\widetilde{Q}}[\exp(\int_{\log(\eta_0/\eta_{\alpha-})}^{\log(\eta_T/\eta_{\alpha-})}(w(z)-w'(z))dz)] := e^{-\alpha\widehat{X}_0}e^{-\alpha\beta T}E_{\widetilde{Q}}[\exp(N_T)],$$

and since N_T, $0 < T < \infty$, is uniformly bounded, we have

$$\liminf_{T\to\infty}\frac{-1}{\alpha T}\log(E[e^{-\alpha\widehat{X}^\varphi_T}]) = \liminf_{T\to\infty}\frac{-1}{\alpha T}(-\alpha\widehat{X}_0 - \alpha\beta T + \log(E_{\widetilde{Q}}[\exp(N_T)])) = \beta$$

in (3). On the other hand, by the Girsanov's theorem,

$$e^{-\alpha\widehat{X}_0 - E_{\widetilde{Q}}[\log Z_T]} = \exp(-\alpha\widehat{X}_0 + E_{\widetilde{Q}}[\int_0^T \sigma w d\widetilde{W}_t - \frac{1}{2}\int_0^T \sigma^2 w^2 dt]) = \cdots$$

where $\widetilde{W}_t = W_t + \int_0^t \sigma w ds$ denotes a $\widetilde{Q}$-Brownian motion, and similarly as in (4), we get

$$\begin{aligned}\cdots &= e^{-\alpha\widehat{X}_0}e^{-\alpha\beta T}\exp(E_{\widetilde{Q}}[\int_{\log(\eta_0/\eta_{\alpha-})}^{\log(\eta_T/\eta_{\alpha-})}(w(z)-w'(z))dz - \int_0^T \sigma(w-w')d\widetilde{W}_t]) \\ &= e^{-\alpha\widehat{X}_0}e^{-\alpha\beta T}\exp(E_{\widetilde{Q}}[\int_{\log(\eta_0/\eta_{\alpha-})}^{\log(\eta_T/\eta_{\alpha-})}(w(z)-w'(z))dz]) = e^{-\alpha\widehat{X}_0}e^{-\alpha\beta T}\exp(E_{\widetilde{Q}}[N_T]).\end{aligned}$$

So,

$$\begin{aligned}\liminf_{T\to\infty}\frac{1}{T}(\widetilde{X}^\varphi_0 + \frac{1}{\alpha}E_{\widetilde{Q}}[\log(Z_T)]) &= \liminf_{T\to\infty}\frac{1}{T}(\widehat{X}_0 + \frac{1}{\alpha}E_{\widetilde{Q}}[\log(Z_T)]) \\ &= \liminf_{T\to\infty}\frac{1}{T}(\beta T - \frac{1}{\alpha}E_{\widetilde{Q}}[N_T]) = \beta.\end{aligned}$$

In the view of (3), $\widehat{\eta}$ is long-term optimal.

Solution 7-5

By the definition, we have $\widehat{\varphi}^0_t = \widehat{X}_t - \widehat{\eta}$, $t \geq 0$, $\widehat{\varphi}^0_{0-} = x$, and $\widehat{\varphi}_t = \widehat{\eta}_t/\widetilde{S}_t$, $t \geq 0$, $\widehat{\varphi}_{0-} = y$. As $\widehat{\varphi}$ only increases (resp. decreases) when $\widetilde{S} = S$ (resp. $\widetilde{S} = (1-\lambda)S$), the strategy $(\widehat{\varphi}^0, \widehat{\varphi})$ is self-financing, and since $(1-\lambda)S \leq \widetilde{S} \leq S$, it is bounded as well, so $\widehat{\varphi} \in \mathcal{A}$. Moreover, since $S \geq \widetilde{S} \geq (1-\lambda)S$ and $0 < \widehat{\varphi} < \eta_{\alpha-}/S$, we have

$$\widehat{\varphi}^0 + \widehat{\varphi}\widetilde{S} \geq \widehat{\varphi} + \widehat{\varphi}^+(1-\lambda)S - \widehat{\varphi}^- S \geq \widehat{\varphi}^0 + \widehat{\varphi}\widetilde{S} - \lambda\eta_{\alpha-},$$

which yields

$$\liminf_{T\to\infty}\frac{-1}{\alpha T}\log(E[e^{-\alpha(\widehat{\varphi}^0_T + \widehat{\varphi}^+_T(1-\lambda)S_T - \widehat{\varphi}^- S_T)}]) = \liminf_{T\to\infty}\frac{-1}{\alpha T}\log(E[e^{-\alpha(\widehat{\varphi}^0_T + \widehat{\varphi}\widetilde{S}_T)}]).$$

Now let (φ^0, φ) be any admissible strategy for the original problem. Set $\widetilde{\varphi}^0_t = \varphi^0_{0-} - \int_0^t \widetilde{S}_t d\varphi_t$. Then (φ^0, φ) is a self-financing strategy for $\widetilde{S}$ with $\widetilde{\varphi}^0 \geq \varphi^0$. We have

$$\begin{aligned}\liminf_{T\to\infty}\frac{-1}{\alpha T}\log(E[e^{-\alpha(\varphi^0_T + \varphi^+_T(1-\lambda)S_T) - \varphi^- S_T)}]) &\leq \liminf_{T\to\infty}\frac{-1}{\alpha T}\log(E[e^{-\alpha(\widetilde{\varphi}^0_T + \varphi_T\widetilde{S}_T)}]) \\ &\leq \liminf_{T\to\infty}\frac{-1}{\alpha T}\log(E[e^{-\alpha(\widehat{\varphi}^0_T + \widehat{\varphi}_T\widetilde{S}_T)}]) \\ &= \liminf_{T\to\infty}\frac{-1}{\alpha T}\log(E[e^{-\alpha(\widehat{\varphi}^0_T + \widehat{\varphi}^+_T(1-\lambda)S_T) - \widehat{\varphi}^- S_T)}]).\end{aligned}$$

We conclude that the strategy $\widehat{\eta}$ is long-term optimal with the equivalent annuity β.