ETH Zürich HS 2015
D-MATH
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Mathematical Finance
Exercise Sheet 2

Please hand in until Friday, 9.10.2015, 12:00 noon, in HG G 53.

Exercise 2-1

Let Θ be the class of self-financing strategies, and Θ_a be the class of admissible self-financing strategies. We call Type 1 arbitrage opportunity an admissible strategy $\theta = (\theta^0, \theta^1, ..., \theta^d) \in \Theta_a$ s.t. $V_0(\theta) = 0$, $V_t(\theta) \geq 0$ a.s. for all $t \in \mathbf{T} := \{0, 1, ..., T\}$ and $P(V_T(\theta) > 0) > 0$, and Type 2 arbitrage opportunity a self-financing strategy $\theta \in \Theta$ s.t. $V_0(\theta) = 0$, $V_T(\theta) \geq 0$ a.s. and $P(V_T(\theta) > 0) > 0$. Show that the existence of Type 2 arbitrage opportunity implies existence of Type 1 arbitrage opportunity.

Exercise 2-2

Assume that the stock price process $S = (S_n)_{n\geq 1}$ is given by

$$S_n = S_{n-1} + \beta_n Y_n,\ n \geq 1,$$
$$S_0 \in \mathbb{R},$$

where Y_n are independent random variables with $P(Y_n = +1) = \frac{1+\alpha_n}{2}$ and $P(Y_n = -1) = \frac{1-\alpha_n}{2}$, and $0 < \alpha_n, \beta_n < 1$, $n \geq 1$.

a) Show that if $\beta_n > \sum_{k=n+1}^{\infty} \beta_k$, for every $n \geq 1$, then S does not admit simple arbitrage.

b) Show that there exists a local martingale measure Q equivalent to P if and only if $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$.

Exercise 2-3

Let μ be a probability measure on $\mathbb{R}_+$ such that $\int x d\mu(x) = 1$. Show that there exists a martingale measure on $\Omega := C_+([0,T])|_{\omega_0=1}$ with a marginal μ at T.

Exercise 2-4

Let $W = (W_t)_{t\in\mathbb{R}_+}$ be a standard $\mathbb{R}^d$-valued Brownian motion, and w a twice continuously differentiable function. Assume that the value of a stock is $S = w(W)$. Show that S is a continuous local martingale for every initial price configuration $S_0 \in \mathbb{R}^d$ if and only if $\Delta w = 0$.

Recall the following definitions. Let μ and υ be probability measures on $\mathbb{R}$. We say that μ is less than υ in the *convex order*, denoted $\leq_{cx}$, if

$$\int \phi d\mu \leq \int \phi d\upsilon$$

for all convex $\phi : \mathbb{R} \to \mathbb{R}_+$. A *coupling* of probability measures μ and υ is a probability measure on $\mathbb{R}^2$ having μ and υ as its marginal distributions.

Exercise 2-5

Show that, for two probability measures μ and υ with finite first moments,

$$\mu \leq_{cx} \upsilon$$

is equivalent to

μ and υ admit a martingale coupling.