ETH Zürich HS 2015
D-MATH
Prof. H. Mete Soner

Coordinator
Matti Kiiski

Mathematical Finance
Solutions Sheet 4

Solution 4-1

Writing $X_t := (\mu - r)t + \sigma W_t + J_t$, we see that

$$dS_t = S_{t-} dX_t.$$

Hence, S is the stochastic exponential given by

$$\begin{aligned} S_t &= S_0 \mathcal{E}(X)_t \\ &= S_0 \exp\left(X_t - \frac{1}{2}[X]_t \right) \prod_{0<s\le t} (1 + \Delta X_s) \exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2 \right) \end{aligned}$$

with $\Delta X_s = X_s - X_{s-}$ and

$$[X]_t = \langle X^c, X^c \rangle_t + \sum_{0<s\le t} (\Delta X_s)^2,$$

where X^c denotes the continuous local martingale part of X. Therefore,

$$\begin{aligned} &\exp\left(X_t - \frac{1}{2}[X]_t \right) \prod_{0<s\le t} \exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2 \right) \\ &= \exp\left(\left(\mu - r - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \right). \end{aligned}$$

On the other hand,

$$\prod_{0<s\le t} (1 + \Delta X_s) = \prod_{i=1}^{N_t} Y_i = \exp\left(\sum_{i=1}^{N_t} \log(Y_i) \right).$$

Then $\tilde{J}_t := \sum_{i=1}^{N_t} \log(Y_i)$ is a compound Poisson process so that

$$\tilde{X}_t = \left(\mu - r - \frac{1}{2}\sigma^2 \right) t + \sigma W_t + \tilde{J}_t$$

is a Levy process.

Solution 4-2

a) $d\langle B, W \rangle_t = \rho dt$ because $d\langle W, W' \rangle_t = 0$ and

$$\begin{aligned} \langle S, Y \rangle_t &= \left\langle \int \sigma(u, S_u, Y_u) dW_u, \int a(u, Y_u) dB_u \right\rangle_t \\ &= \int_0^t \sigma(u, S_u, Y_u) a(u, Y_u) d\langle W, B \rangle_u = \int_0^t \sigma(u, S_u, Y_u) a(u, Y_u) \rho du. \end{aligned}$$

b) Following the lecture notes $Q \sim P$ has a density process Z^Q which has a continuous version. Defining L^Q by

$$L^Q = \int \frac{1}{Z^Q} dZ^Q$$

we have $Z^Q = Z_0^Q \mathcal{E}(L^Q)$. By a general version of Kunita-Watanabe decomposition L^Q is in our case given by

$$L^Q = \int \gamma^Q \sigma dW + N^Q$$

with $N^Q \in \mathcal{M}_{0,loc}(P)$ and $\langle N^Q, \int \sigma dW \rangle = 0$. By Bayes' rule Q is an ELMM for S iff $Z^Q S \in M_{loc}(P)$. Thus by the product rule we obtain

$$\begin{aligned} d(Z_t^Q S_t) &= Z_t^Q \sigma_t dW_t + S_t dZ_t^Q + Z_t^Q \left(\mu_t dt + d\left\langle L^Q, \int \sigma dW \right\rangle_t \right) \\ &= Z_t^Q \sigma_t dW_t + S_t dZ_t^Q + Z_t^Q \left(\mu_t dt + \gamma^Q \sigma^2 dt \right), \end{aligned}$$

yielding $Z^Q S \in \mathcal{M}_{loc}(P)$ if and only if $\gamma^Q = -\frac{\mu_t}{\sigma_t^2}$. Therefore the equivalent local martingale measures Q are parametrized via

$$\frac{Z^Q}{Z_0^Q} = \mathcal{E}\left(-\int \frac{\mu}{\sigma} dW + N^Q \right).$$

Since the filtration is generated by (W, W') we can apply the martingale representation theorem to write N^Q as

$$N^Q = \int \psi dW + \int \nu dW',$$

where ψ and ν are some predictable processes. As $\langle N^Q, \int \sigma dW \rangle = 0$, it follows that $\psi = 0$ such that we finally obtain

$$\frac{Z^Q}{Z_0^Q} = \mathcal{E}\left(-\int \lambda dW + \int \nu dW' \right)$$

where $\lambda = \mu/\sigma$ and ν is some predictable process.

c) We want to change the measure, so we must assume that ν is sufficiently nice, for instance that it satisfies Novikov's condition

$$E\left[e^{\frac{1}{2}\int_0^T \nu_t^2 dt} \right] < \infty.$$

By Girsanov, (W^Q, W'^Q), defined by $W^Q = W + \int \lambda dt$ and $W'^Q = W' - \int \nu dt$ is a 2-dimensional Q-Brownian motion. Plugging this into the SDE's

$$dS = \mu dt + \sigma(dW^Q - \lambda dt) = (\mu - \lambda\sigma)dt + \sigma dW^Q$$

and

$$\begin{aligned} dY &= bdt + a\rho(dW^Q - \lambda dt) + a\sqrt{1-\rho^2}(dW'^Q + \nu dt) \\ &= (b + a(\sqrt{1-\rho^2}\nu - \rho\lambda))dt + adB^Q \end{aligned}$$

for the Q-Brownian motion $B^Q = \rho W_t^Q + \sqrt{1-\rho^2} W_t'^Q$.

Solution 4-3

a) To show convexity of $\mathcal{C}(x)$, let $X^1, X^2 \in \mathcal{C}(x)$ and $\lambda \in]0,1[$. Then,

$$\begin{aligned}\sup_{Y\in\mathcal{D}} E[Y(\lambda X^1 + (1-\lambda)X^2)] &\le \lambda \sup_{Y\in\mathcal{D}} \mathbb{E}[YX^1] + (1-\lambda)\sup_{Y\in\mathcal{D}} \mathbb{E}[YX^2] \\ &\le \lambda x + (1-\lambda)x = x,\end{aligned}$$

and so $\lambda X^1 + (1-\lambda)X^2 \in \mathcal{C}(x)$. To show closedness in $L^0_+(\mathcal{F}_T)$, let $(X^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}(x)$ converging in probability to some $X \in L^0_+(\mathcal{F}_T)$. Then a subsequence, called again $(X^n)_{n\in\mathbb{N}}$, converges $P-$a.s. to X. Hence, for each $Y \in \mathcal{D}$, by Fatou's lemma,

$$\mathbb{E}[YX] \le \liminf_{n\to\infty} \mathbb{E}[YX^n] \le x.$$

Since $\mathcal{D}$ contains strictly positive random variables (e.g. the densities of equivalent σ-martingale measures), this implies that $X \in L^0(\mathcal{F}_T)$. Now, taking the supremum over $\mathcal{D}$ gives $X \in \mathcal{C}(x)$.

b) It suffices to show that $E[U(\cdot)]$ is continuous from below, i.e., that $EU(X^n) \uparrow EU(X)$ as $X^n \uparrow X$ in P in $\mathcal{C}(x)$. Without loss of generality, we may assume that $U^-(X^n) \in L^1(P)$ for all $n \in \mathbb{N}$. By (a), $\mathcal{C}(x)$ is convex and closed, and since $P << Q$ and

$$\lim_{n\to\infty} \sup_{X_T\in\mathcal{C}(x)} Q(|X_T| > n) \le \lim_{n\to\infty} \sup_{H\in\mathcal{A}} Q(|(H\cdot S)_T| > n) = 0,$$

it is also bounded. So, we may apply Komlos' lemma (see e.g. W. Schahermayer, Optimal Investment in Incomplete Financial Markets, Lemma 3.3.): there exists a sequence $(\widetilde{X}^n)_{n\in\mathbb{N}}$ with $\widetilde{X}^n \in \operatorname{conv}(X^n, X^{n+1}, \ldots)$ such that $\widetilde{X}^n$ converges P-a.s., and a fortiori in probability, to some $\mathcal{F}_T$-measurable random variable $\widehat{X}$ taking values in $[0,\infty]$. By convexity of $\mathcal{C}(x)$, $\widetilde{X}^n \in \mathcal{C}(x)$ for all $n \in \mathbb{N}$, and by closedness of $\mathcal{C}(x)$ in $L^0_+(\mathcal{F}_T)$, $\widehat{X} \in \mathcal{C}(x)$.

Now, uniform integrability of $U^+(\mathcal{C}(x))$ (which follows from the assumption $u(x) < \infty$) gives

$$\lim_{n\to\infty} E[U^+(\widetilde{X}^n)] = E[U^+(\widehat{X})],$$

while Fatou's lemma gives

$$\liminf_{n\to\infty} E[U^-(\widetilde{X}^n)] \ge E[U^-(\widehat{X})].$$

Moreover, since $E[U(\cdot)]$ is concave and increasing and $(X^n)_{n\in\mathbb{N}}$ is increasing, we have

$$\begin{aligned}E[U(\widetilde{X}^n)] = E[U(\lambda_1^n X^n + \lambda_2^n X^{n+1}\cdots)] &\ge \lambda_1^n E[U(X^n)] + \lambda_2^n E[U(X^{n+1})\cdots \\ &\ge \inf_{m\ge n} E[U(X^m)] = E[U(X^n)], \quad n \in \mathbb{N}.\end{aligned}$$

Putting everything together gives

$$E[U(\widehat{X})] \ge \liminf_{n\to\infty} E[U(\widetilde{X}^n)] \ge \liminf_{n\to\infty} E[U(X^n)] = E[U(x)].$$

In particular, if we choose $X^n \in \{X \in \mathcal{C}(x) : 1/n \le u(x) - E[U(X)] \le 1/(n-1)\}$, $n \in \mathbb{N}$, we see that $u(x)$ is attained for some $\widehat{X} \in \mathcal{C}(x)$.

c) Choose $r > 1$ small enough that $p := \frac{1}{rb} > 1$. We show that $U^+(\mathcal{C}(x))$ is bounded in $L^r(P)$ and hence uniformly integrable. To this end, by the assumption on U^+, it suffices to show that

$$\sup_{X \in \mathcal{C}(x)} E[(X^b)^r] = \sup_{X \in \mathcal{C}(x)} E[X^{1/p}] < \infty.$$

Let $q > 1$ be the exponent conjugate to p. Then, by Hölder's inequality, by the hint of part (a), and by the fact that $\left(\frac{dQ}{dP}\right)^{-1}$ has moments of all orders, for each $X \in \mathcal{C}(x)$,

$$E[X^{1/p}] = E\left[\left(X\frac{dQ}{dP}\right)^{1/p}\left(\frac{dQ}{dP}\right)^{-1/p}\right] \leq \left(E\left[X\frac{dQ}{dP}\right]\right)^{1/p}\left(E\left[\left(\frac{dQ}{dP}\right)^{-q/p}\right]\right)^{1/q}$$
$$\leq x^{1/p}\left(E\left[\left(\frac{dQ}{dP}\right)^{-q/p}\right]\right)^{1/q} < \infty.$$

Taking the supremum over $\mathcal{C}(x)$ establishes the claim.

Solution 4-4

Denote by $Z = (Z_t)_{t\in[0,T]}$ the density process of Q with respect to P.

a) The second claim follows immediately from the first claim together with the fact that $yZ_T = y\frac{dQ}{dP} \in \mathcal{D}(y)$ and the fact that the function V is decreasing. So it remains to show the first claim. Seeking a contradiction, suppose there exists $z \in \mathcal{D}(y)$ such that $A := \{z > yZ_T\}$ has $P[A] > 0$. Set $a = Q[A] > 0$ and define the Q-martingale $M = (M_t)_{t\in[0,T]}$ by $M_t := E_Q[1_A \mid \mathcal{F}_t]$. Then M is nonnegative and $M_0 = a$ by the fact that F_0 is P-trivial. Under Q, there exists admissible H such that $M = a + H \cdot S$. It follows that $M_T \in \mathcal{C}(a)$, i.e., $\frac{1}{a}M_T \in \mathcal{C}(1)$. Now, on the one hand, by the definition of $\mathcal{D}(y)$,

$$E[\frac{1}{a}M_T z] \leq y \text{ i.e. } E[M_T z] \leq ay.$$

On the other hand,

$$E[M_T Z_T] = E_Q[M_T] = M_0 = a.$$

Thus, we arrive at the contradiction

$$0 \geq E[M_T(z - yZ_T)] = E[1_{\{z>yZ_T\}}(z - yZ_T)] > 0.$$

b) Note that $0 \leq y_0 < \infty$ and $v(y) < \infty$ on $]y_0, \infty[$. Moreover, recall that the function V is strictly decreasing, strictly convex and continuous on $]0, \infty[$.

First, define the function $g :]y_0, \infty[\to [-\infty, 0]$ by

$$g(s) = E[Z_T V'(sZ_T)].$$

This is well defined as $Z_T \geq 0$ P-a.s. and $V' < 0$. Moreover, it is increasing as V' is increasing. Thus if $g(s_0) > -\infty$ for some $s_0 > y_0$, it follows by dominated convergence that it is continuous on $[s_0, \infty)$.

Next, for $y_1, y_2 \in]y_0, \infty[$, $y_1 < y_2$, the fundamental theorem of calculus gives

$$V(y_1 Z_T) - V(y_2 Z_T) = \int_{y_1}^{y_2} Z_T V'(sZ_T) ds. \qquad (1)$$

Now, the left hand side of (1) is integrable by assumption. Thus, the right hand side is so, too, and since $V' < 0$, the integrand on the right hand side is strictly negative, and Fubini's theorem gives

$$v(y_2) - v(y_1) = \int_{y_1}^{y_2} g(s) ds.$$

In particular, the function g is finite a.e. on $]y_0, \infty[$, and thus continuous and finite on $]y_0, \infty[$. Now the claim follows from the fundamental theorem of calculus.

c) First, by the hint of part (a) of the previous exercise $X \in \mathcal{C}(x)$ if and only if

$$\sup_{Y \in \mathcal{D}} E[XY] \leq x.$$

By part (a) of this exercise, this is equivalent to

$$\mathbb{E}[XZ_T] \leq x.$$

Now, by part (b) and the choice of y_x,

$$\mathbb{E}[\widehat{X} Z_T] = E[-V'(y_x Z_T) Z_T] = -v'(y_x) = x, \text{ i.e. } U'(\widehat{X}) = y_x Z_T. \tag{2}$$

We see that $\widehat{X} \in \mathcal{C}(x)$.

Next, fix $X \in \mathcal{C}(x)$. We may assume without loss of generality that $E[U(X)] > -\infty$. By the fact that $\widehat{X} > 0$ P-a.s. and U is in $C^1(]0, \infty[)$ and strictly concave on $]0, \infty[$,

$$U(X) - U(\widehat{X}) \leq U'(\widehat{X})(X - \widehat{X}),$$

where the equality is strict on $\{X \neq \widehat{X}\}$. Taking expectations, (2) yields

$$\mathbb{E}[U(X) - U(\widehat{X})] \leq \mathbb{E}[U'(\widehat{X})(X - \widehat{X})] = y_x \mathbb{E}[Z_T(X - \widehat{X})] \leq 0.$$

If $X = \widehat{X}$ P-a.s., then both inequalities are trivially equalities, and if $P[X \neq \widehat{X}] > 0$, then the first inequality is strict.

Solution 4-5

The discounted stock price process S^1 satisfies the SDE

$$dS_t^1 = S_t^1((\mu - r)dt + \sigma dW_t) = S_t^1 \sigma(\lambda dt + dW_t),$$

where $\lambda := \frac{\mu - r}{\sigma}$ denotes the market price of risk. By the previous round exercise 3-1, there exists a unique equivalent martingale measure $Q \sim P$ on $\mathcal{F}_T$ given by

$$\frac{dQ}{dP} = \mathcal{E}(-\lambda W)_T.$$

Moreover, elementary analysis gives $V(y) = \frac{1-\gamma}{\gamma} y^{-\frac{\gamma}{1-\gamma}}$ and $V'(y) = -y^{-\frac{1}{1-\gamma}}$.

a) Fix $y > 0$. Then by Exercise 4-4 (a) and the fact that $\mathcal{E}(aW)$ is a P-martingale for all $a \in \mathbb{R}$,

$$\begin{aligned}
v(y) &= E\left[\frac{1-\gamma}{\gamma} y^{-\frac{\gamma}{1-\gamma}} \left(\mathcal{E}(-\lambda W)_T\right)^{-\frac{\gamma}{1-\gamma}}\right] \\
&= \frac{1-\gamma}{\gamma} y^{-\frac{\gamma}{1-\gamma}} E\left[\exp\left(\frac{\lambda\gamma}{1-\gamma} W_T + \frac{1}{2}\frac{\lambda^2\gamma}{1-\gamma} T\right)\right] \\
&= \frac{1-\gamma}{\gamma} y^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2} T\right) E\left[\mathcal{E}\left(\frac{\lambda\gamma}{1-\gamma} W\right)_T\right] \\
&= \frac{1-\gamma}{\gamma} y^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2} T\right) < \infty.
\end{aligned}$$

b) First, note that $v(y) < \infty$ for some $y \in]0, \infty[$ implies that

$$u(x) \leq v(y) + vx < \infty, \quad x \in]0, \infty[.$$

Next, fix $x > 0$. Then by Exercise 4-4 (b) and part (a),

$$\begin{aligned}
\widehat{X}_x &= -V'\left(y_x \frac{dQ}{dP}\right) = y_x^{-\frac{1}{1-\gamma}} \left(\mathcal{E}(-\lambda W)_T\right)^{-\frac{1}{1-\gamma}} \\
&= -v'(y_x) \exp\left(-\frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2}T\right) \exp\left(\frac{\lambda}{1-\gamma}W_T + \frac{1}{2}\frac{\lambda^2}{1-\gamma}T\right) \\
&= x \exp\left(\frac{\lambda}{1-\gamma}(W_T + \lambda T) - \frac{1}{2}\frac{\lambda^2}{(1-\gamma)^2}T\right) \\
&= x\mathcal{E}\left(\frac{\lambda}{1-\gamma}R\right)_T.
\end{aligned}$$

c) Fix $x > 0$. By the definition of the stochastic exponential,

$$\begin{aligned}
\widehat{X} &= x\left(1 + \int_0^T \mathcal{E}\left(\frac{\lambda}{1-\gamma}R\right)_t \frac{\lambda}{1-\gamma} dR_t\right) \\
&= x + \int_0^T x\mathcal{E}\left(\frac{\lambda}{1-\gamma}R\right)_t \frac{\lambda}{1-\gamma}\frac{1}{\sigma S_t} dS_t.
\end{aligned}$$

This gives the first claim. Using again that $\mathcal{E}(aW)$ is a P-martingale for all $a \in \mathbb{R}$ gives

$$\begin{aligned}
u(x) = \mathbb{E}\left[U(\widehat{X}_x)\right] &= \frac{x^\gamma}{\gamma}\mathbb{E}\left[\left(\mathcal{E}\left(\frac{\lambda}{1-\gamma}R\right)_T\right)^\gamma\right] \\
&= \frac{x^\gamma}{\gamma}\mathbb{E}\left[\exp\left(\frac{\lambda\gamma}{1-\gamma}(W_T + \lambda T) - \frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2}T\right)\right] \\
&= \frac{x^\gamma}{\gamma}\exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{1-\gamma}T\right)\mathbb{E}\left[\mathcal{E}\left(\frac{\lambda\gamma}{1-\gamma}W\right)_T\right] \\
&= \frac{x^\gamma}{\gamma}\exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{1-\gamma}T\right).
\end{aligned}$$

This establishes the second claim.