Musterlösung 12

1. **a)** Wir benutzen partielle Integration:

$$\begin{aligned}\int_{1-\varepsilon}^{1} \frac{3}{2}\sqrt{x}\ln x\,dx &= x^{\frac{3}{2}}\ln x\Big|_{x=1-\varepsilon}^{x=1} - \int_{1-\varepsilon}^{1} x^{\frac{3}{2}}\frac{1}{x}\,dx \\ &= -(1-\varepsilon)^{3/2}\ln(1-\varepsilon) - \int_{1-\varepsilon}^{1}\sqrt{x}\,dx \\ &= -(1-\varepsilon)^{3/2}\ln(1-\varepsilon) - \frac{2}{3}\,x^{\frac{3}{2}}\Big|_{x=1-\varepsilon}^{x=1} \\ &= -(1-\varepsilon)^{3/2}\ln(1-\varepsilon) - \frac{2}{3} + \frac{2}{3}(1-\varepsilon)^{3/2}\,.\end{aligned}$$

b) Wir machen die Substitution $x = e^t$ und bekommen $dx = e^t dt$, $x(1) = e$ und $x(2) = e^2$. Daher gilt

$$\begin{aligned}\int_1^2 \frac{e^t(e^t-1)}{e^{2t}-1}\,dt &= \int_e^{e^2} \frac{x-1}{x^2-1}\,dx \\ &= \int_e^{e^2} \frac{1}{x+1}\,dx \\ &= \log|x+1|\big|_e^{e^2} \\ &= \log\left(\frac{e^2+1}{e+1}\right)\,.\end{aligned}$$

c) Nach geschicktem Erweitern des Integranden mit $\cos x$ ist der Zähler gerade das (-1)-fache der Ableitung des Nenners:

$$\begin{aligned}\int_0^{\pi/4} \frac{2\sin x + \tan x}{1+\cos x}\,dx &= \int_0^{\pi/4} \frac{2\sin x\cos x + \sin x}{\cos x + \cos^2 x}\,dx \\ &= -\ln(\cos x + \cos^2 x)\Big|_{x=0}^{x=\pi/4} \\ &= -\ln\left(\frac{\sqrt{2}}{2} + \frac{1}{2}\right) + \ln(1+1) \\ &= \ln\frac{4}{\sqrt{2}+1}\end{aligned}$$

Bitte wenden!

2. Offensichtlich gilt $\boxed{I(p,q) = I(q,p)}$ (substituiere $t = 1 - x$).

$$I(p,q) = \int\limits_0^1 x^{p+1}\,(1-x)^q\,dx + \int\limits_0^1 (x^p - x^{p+1})(1-x)^q\,dx = I(p+1,q) + I(p,q+1)$$

$$\begin{aligned}
I(p+1,q) &= \int\limits_0^1 x^{p+1}\,(1-x)^q\,dx \\
&= \left[-\frac{1}{q+1}\,x^{p+1}\,(1-x)^{q+1}\right]_0^1 + \int\limits_0^1 \frac{p+1}{q+1}\,x^p\,(1-x)^{q+1}\,dx \\
&= \frac{p+1}{q+1}\,I(p,q+1) = \frac{p+1}{q+1}\big(I(p,q) - I(p+1,q)\big)
\end{aligned}$$

Wir können nun die letzte Gleichung nach $I(p+1,q)$ auflösen. Es folgt

$$\boxed{\begin{array}{l} I(p+1,q) = \dfrac{p+1}{p+q+2}\,I(p,q) \\ I(0,q) = I(q,0) = \dfrac{1}{q+1}\,,\ \text{insbesondere } I(0,0) = 1 \end{array}}$$

Es gilt

$$\boxed{I(p,q) = \frac{p!\,q!}{(p+q+1)!}}$$

3. Zuerst muss man bestimmen, wo $K_1(x) \geq K_2(x)$ bzw. $K_2(x) \geq K_1(x)$ gilt. Äquivalent dazu fragt man sich, wo $\sin x \geq \cos x$ bzw. $\cos x \geq \sin x$ ist. Man erhält:

$$\sin x \geq \cos x \Leftrightarrow x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]\,,$$

$$\cos x \geq \sin x \Leftrightarrow x \in \left[0, \frac{\pi}{4}\right] \cup \left[\frac{5\pi}{4}, 2\pi\right]\,.$$

Deshalb folgt

$$\begin{aligned}
F &= \int_0^{\pi/4} (3 + \cos x - 3 - \sin x)\,dx + \int_{\pi/4}^{5\pi/4} (3 + \sin x - 3 - \cos x)\,dx + \\
&+ \int_{5\pi/4}^{2\pi} (3 + \cos x - 3 - \sin x)\,dx \\
&= (\cos x + \sin x)|_0^{\pi/4} - (\cos x + \sin x)|_{\pi/4}^{5\pi/4} + (\cos x + \sin x)|_{5\pi/4}^{2\pi} = \\
&= 4\sqrt{2}\,.
\end{aligned}$$

Siehe nächstes Blatt!

4. Wir setzen $t = \sqrt{x}$ und bekommen $x = t^2$ und $dx = 2tdt$. Dann es folgt

$$\begin{aligned}\int \frac{1}{\sqrt{x}+1}\,dx &= 2\left(\int \frac{t}{t+1}\,dt\right)_{t=\sqrt{x}} = 2\left(\int dt - \int \frac{1}{t+1}\,dt\right)_{t=\sqrt{x}} \\ &= 2(t - \log|t+1|)_{t=\sqrt{x}} + C = 2(\sqrt{x} - \log(\sqrt{x}+1)) + C\,.\end{aligned}$$

Wir setzen die Bedingung

$$(2\sqrt{x} - 2\log(\sqrt{x}+1) + C)_{x=0} = 1$$

ein und erhalten somit $C = 1$. Es folgt

$$K(x) = 2\sqrt{x} - 2\log(\sqrt{x}+1) + 1\,.$$

Bitte wenden!

English version

1. **a)** We use integration by parts:

$$\begin{aligned}\int_{1-\varepsilon}^{1} \frac{3}{2}\sqrt{x}\ln x\,dx &= x^{\frac{3}{2}}\ln x\Big|_{x=1-\varepsilon}^{x=1} - \int_{1-\varepsilon}^{1} x^{\frac{3}{2}}\frac{1}{x}\,dx \\ &= -(1-\varepsilon)^{3/2}\ln(1-\varepsilon) - \int_{1-\varepsilon}^{1}\sqrt{x}\,dx \\ &= -(1-\varepsilon)^{3/2}\ln(1-\varepsilon) - \frac{2}{3}\,x^{\frac{3}{2}}\Big|_{x=1-\varepsilon}^{x=1} \\ &= -(1-\varepsilon)^{3/2}\ln(1-\varepsilon) - \frac{2}{3} + \frac{2}{3}(1-\varepsilon)^{3/2}\,.\end{aligned}$$

b) We do the substitution $x = e^t$ and obtain $dx = e^t dt$, $x(1) = e$ and $x(2) = e^2$. Hence, we have

$$\begin{aligned}\int_1^2 \frac{e^t(e^t-1)}{e^{2t}-1}\,dt &= \int_e^{e^2}\frac{x-1}{x^2-1}\,dx \\ &= \int_e^{e^2}\frac{1}{x+1}\,dx \\ &= \log|x+1|\big|_e^{e^2} \\ &= \log\left(\frac{e^2+1}{e+1}\right)\,.\end{aligned}$$

c) We multiply with $\cos x$ and get:

$$\begin{aligned}\int_0^{\pi/4}\frac{2\sin x+\tan x}{1+\cos x}\,dx &= \int_0^{\pi/4}\frac{2\sin x\cos x+\sin x}{\cos x+\cos^2 x}\,dx \\ &= -\ln(\cos x+\cos^2 x)\Big|_{x=0}^{x=\pi/4} \\ &= -\ln\left(\frac{\sqrt{2}}{2}+\frac{1}{2}\right)+\ln(1+1) \\ &= \ln\frac{4}{\sqrt{2}+1}\end{aligned}$$

Siehe nächstes Blatt!

2. Obviously, it is $\boxed{I(p,q) = I(q,p)}$ (substitute $t = 1 - x$).

$$I(p,q) = \int_0^1 x^{p+1}\,(1-x)^q\,dx + \int_0^1 (x^p - x^{p+1})(1-x)^q\,dx = I(p+1,q) + I(p,q+1)$$

$$\begin{aligned} I(p+1,q) &= \int_0^1 x^{p+1}\,(1-x)^q\,dx \\ &= \left[-\frac{1}{q+1}\,x^{p+1}\,(1-x)^{q+1}\right]_0^1 + \int_0^1 \frac{p+1}{q+1}\,x^p\,(1-x)^{q+1}\,dx \\ &= \frac{p+1}{q+1}\,I(p,q+1) = \frac{p+1}{q+1}\big(I(p,q) - I(p+1,q)\big) \end{aligned}$$

We may solve the last equation with respect to $I(p+1,q)$. It follows

$$\boxed{\begin{array}{l} I(p+1,q) = \dfrac{p+1}{p+q+2}\,I(p,q) \\ I(0,q) = I(q,0) = \dfrac{1}{q+1}\,,\ \text{in particular } I(0,0) = 1 \end{array}}$$

We conclude

$$\boxed{I(p,q) = \frac{p!\,q!}{(p+q+1)!}}$$

3. First of all, we must determine where $K_1(x) \geq K_2(x)$ respectively $K_2(x) \geq K_1(x)$ happens. That is equivalent to ask where $\sin x \geq \cos x$ respectively $\cos x \geq \sin x$. One gets:

$$\sin x \geq \cos x \Leftrightarrow x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]\,,$$

$$\cos x \geq \sin x \Leftrightarrow x \in \left[0, \frac{\pi}{4}\right] \cup \left[\frac{5\pi}{4}, 2\pi\right]\,.$$

Hence, it follows

$$\begin{aligned} F &= \int_0^{\pi/4} (3 + \cos x - 3 - \sin x)\,dx + \int_{\pi/4}^{5\pi/4} (3 + \sin x - 3 - \cos x)\,dx + \\ &+ \int_{5\pi/4}^{2\pi} (3 + \cos x - 3 - \sin x)\,dx \\ &= (\cos x + \sin x)|_0^{\pi/4} - (\cos x + \sin x)|_{\pi/4}^{5\pi/4} + (\cos x + \sin x)|_{5\pi/4}^{2\pi} = \\ &= 4\sqrt{2}\,. \end{aligned}$$

Bitte wenden!

4. We set $t = \sqrt{x}$ and obtain $x = t^2$ and $dx = 2tdt$. Then it follows

$$\begin{aligned} \int \frac{1}{\sqrt{x}+1}\,dx &= 2\left(\int \frac{t}{t+1}\,dt\right)_{t=\sqrt{x}} = 2\left(\int dt - \int \frac{1}{t+1}\,dt\right)_{t=\sqrt{x}} \\ &= 2(t - \log|t+1|)_{t=\sqrt{x}} + C = 2(\sqrt{x} - \log(\sqrt{x}+1)) + C\,. \end{aligned}$$

We impose the condition

$$(2\sqrt{x} - 2\log(\sqrt{x}+1) + C)_{x=0} = 1$$

and get accordingly $C = 1$. It follows

$$K(x) = 2\sqrt{x} - 2\log(\sqrt{x}+1) + 1\,.$$