Musterlösung 13

1. **a)** Mittels Partialbruchzerlegung lässt sich der Integrand in eine Summe umschreiben, deren einzelne Summanden einfach zu integrieren sind:

$$\frac{1}{x^3 - x} = \frac{1}{x(x+1)(x-1)} \overset{\text{(PBZ)}}{=} -\frac{1}{x} + \frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}\,. \qquad (1)$$

In der Tat machen wir den Ansatz

$$\frac{1}{x^3 - x} = \frac{1}{x(x+1)(x-1)} = \frac{\omega_1}{x} + \frac{\omega_2}{x+1} + \frac{\omega_3}{x-1}$$

und erhalten

$$\omega_1(x^2 - 1) + \omega_2 x(x-1) + \omega_3 x(x+1) = 1\,,$$

d.h.

$$\begin{cases} \omega_1 + \omega_2 + \omega_3 = 0\,, \\ -\omega_2 + \omega_3 = 0\,, \\ -\omega_1 = 1\,. \end{cases}$$

Die Lösungen des Systems sind

$$\omega_1 = -1\,,\ \ \omega_2 = \frac{1}{2}\,,\ \ \omega_3 = \frac{1}{2}\,,$$

und deshalb erhalten wir (1).

$$\begin{aligned} \int_2^3 \frac{dx}{x^3 - x} &= -\int_2^3 \frac{1}{x}\,dx + \frac{1}{2}\int_2^3 \frac{1}{x+1}\,dx + \frac{1}{2}\int_2^3 \frac{1}{x-1}\,dx \\ &= -\ln x\Big|_{x=2}^{x=3} + \frac{1}{2}\ln(x+1)\Big|_{x=2}^{x=3} + \frac{1}{2}\ln(x-1)\Big|_{x=2}^{x=3} \\ &= -\ln\frac{3}{2} + \frac{1}{2}\ln\frac{4}{3} + \frac{1}{2}\ln 2 \\ &= \ln\frac{2}{3} + \ln\frac{2}{\sqrt{3}} + \ln\sqrt{2} = \ln\left(\frac{2}{3}\cdot\frac{2}{\sqrt{3}}\cdot\sqrt{2}\right) \\ &= \ln\sqrt{\frac{32}{27}} \end{aligned}$$

b) Wir verwenden eine Partialbruchzerlegung: Für den Nenner gilt

$$x^2 - 2x - 63 = (x+7)(x-9)$$

und daher bestimmen wir A, B so dass

$$\frac{4x-2}{(x+7)(x-9)} = \frac{A}{x-9} + \frac{B}{x+7}.$$

Wir erhalten

$$4x - 2 = A(x+7) + B(x-9).$$

Einsetzen von $x = -7$, bzw. $x = 9$ liefert

$$-30 = -16B \quad \Rightarrow B = 15/8$$

und

$$34 = 16 \cdot A \quad \Rightarrow A = 17/8$$

also

$$\int \frac{4x-2}{x^2-2x-63}dx = \int \frac{17/8}{x-9}dx + \int \frac{15/8}{x+7}dx = \frac{17}{8}\log(|x-9|) + \frac{15}{8}\log(|x+7|) + c.$$

c) Wir verwenden eine Partialbruchzerlegung mit dem Ansatz

$$\frac{2x+1}{(x+2)^2} = \frac{a}{x+2} + \frac{b}{(x+2)^2}\,.$$

Das liefert nun

$$\frac{2x+1}{(x+2)^2} = \frac{ax+2a+b}{(x+2)^2}$$

und somit das System

$$\begin{cases} a = 2 \\ 2a + b = 1\,. \end{cases}$$

d.h. $a = 2$ und $b = -3$. Wir bekommen deshalb

$$\begin{aligned} \int \frac{2x+1}{(x+2)^2}\,dx &= \int \frac{2}{x+2}\,dx - \int \frac{3}{(x+2)^2}\,dx \\ &= 2\log|x+2| + \frac{3}{x+2} + C \end{aligned}$$

und sind fertig.

Siehe nächstes Blatt!

2. **a)** Wir faktorisieren den Nenner und bekommen

$$(x^2-9)^2 = [(x-3)(x+3)]^2 = (x-3)^2(x+3)^2 \,.$$

Weil die Nullstellen 3 und -3 Multiplizität 2 besitzen, folgt, dass wir den Ansatz

$$\frac{x^2}{(x^2-9)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2}$$

machen müssen, wobei $A, B, C, D \in \mathbb{R}$ zu bestimmen sind.

Somit erhalten wir

$$\frac{A(x-3)(x+3)^2 + B(x+3)^2 + C(x-3)^2(x+3) + D(x-3)^2}{(x-3)^2(x+3)^2} = \frac{x^2}{(x-3)^2(x+3)^2}$$

und

$$(A+C)x^3+(3A+B-3C+D)x^2+(-9A+6B-9C-6D)x+(-27A+9B+27C+9D) = x^2 \,,$$

was uns das System

$$\begin{cases} A+C=0 \\ 3A+B-3C+D=1 \\ -9A+6B-9C-6D=0 \\ -27A+9B+27C+9D=0 \end{cases}$$

liefert. Das System besitzt die Lösungen

$$A=\frac{1}{12},\; B=\frac{1}{4},\; C=-\frac{1}{12},\; D=\frac{1}{4}$$

und deshalb ist die gesuchte Partialbruchzerlegung

$$\frac{x^2}{(x^2-9)^2} = \frac{1}{12(x-3)} + \frac{1}{4(x-3)^2} - \frac{1}{12(x+3)} + \frac{1}{4(x+3)^2} \,.$$

b) Zunächst zerlegen wir die gegebene rationale Funktion durch Polynomdivion in die Summe eines Polynoms $p(x)$ und eines rationalen Anteils $r(x)$, so dass der Grad des Zählers von $r(x)$ kleiner ist als der Grad des Nenners. Wir erhalten

$$(x^{10}-x^7+3x) : (x^3-1) = x^7 + \frac{3x}{x^3-1} \,.$$

Wir faktorisieren den Nenner von $r(x) = \frac{3x}{x^3-1}$ und bekommen

$$r(x) = \frac{3x}{(x-1)(x^2+x+1)} \,.$$

Bitte wenden!

In diesem Fall wird der Ansatz durch

$$\frac{3x}{x^3-1} = \frac{a}{x-1} + \frac{bx+c}{x^2+x+1}$$

gegeben. Somit erhalten wir

$$\frac{3x}{x^3-1} = \frac{ax^2+ax+a+bx^2-bx+cx-c}{x^3-1}$$

und das System

$$\begin{cases} a+b=0 \\ a-b+c=3 \\ a-c=0\,, \end{cases}$$

dessen Lösungen

$$a=1,\ b=-1,\ c=1$$

sind. Die gesuchte Partialbruchzerlegung ist somit

$$\frac{x^{10}-x^7+3x}{x^3-1} = x^7 + \frac{1}{x-1} + \frac{1-x}{x^2+x+1}\,.$$

3. **a)** Die Substitution $t = \sin x$ liefert

$$\int \frac{\cos^3 x}{\sin^4 x}\,dx = \int \frac{(1-\sin^2 x)\cos x}{\sin^4 x}\,dx = \int \frac{1-t^2}{t^4}\,dt = \int (\frac{1}{t^4} - \frac{1}{t^2})\,dt$$
$$= -\frac{1}{3t^3} + \frac{1}{t} + K = -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + K.$$

b) Die Substitution $x = 4\sinh z$ ergibt $dx = 4\cosh z\,dz$ und folglich

$$\begin{aligned}
\int \sqrt{x^2+16}\,dx &= \int \sqrt{16\sinh^2 z + 16}\cdot 4\cosh z\,dz = 16\int \cosh^2 z\,dz \\
&= 8\int (\cosh 2z + 1)\,dz = 4\sinh 2z + 8z + C \\
&= 8\sinh z\cosh z + 8z + C \\
&= 2x\cosh\big(\operatorname{Arsinh}\frac{x}{4}\big) + 8\operatorname{Arsinh}\tfrac{x}{4} + C \\
&= 2x\sqrt{\Big(\frac{x}{4}\Big)^2 + 1} + 8\operatorname{Arsinh}\frac{x}{4} + C \\
&= \frac{1}{2}\,x\sqrt{x^2+16} + 8\operatorname{Arsinh}\frac{x}{4} + C.
\end{aligned}$$

Dabei haben wir die Relation

$$\cosh^2 z = \frac{1}{2}\,(\cosh(2z)+1)$$

Siehe nächstes Blatt!

verwendet.

Bemerkung: Alternativ könnte man das obige Integral $I := \int \cosh^2 z \, dz$ auch mithilfe einer partiellen Integration berechnen:

$$\begin{aligned} I = \int \cosh^2 z \, dz &= \sinh z \cosh z - \int \sinh^2 z \, dz \\ &= \sinh z \cosh z - \int (\cosh^2 z - 1) \, dz = \sinh z \cosh z - \int \cosh^2 z \, dz + z \\ &= \sinh z \cosh z + z - I, \end{aligned}$$

also

$$I = \frac{1}{2} (\sinh z \cosh z + z) \, .$$

c) Diesem Integral begegnen wir mit einer Partialbruchzerlegung:

$$\frac{1}{x^2 - 7x + 10} = \frac{1}{(x-5)(x-2)} \stackrel{!}{=} \frac{A}{x-5} + \frac{B}{x-2}.$$

Multiplikation mit $(x-5)(x-2)$ liefert

$$1 = (x-2)A + (x-5)B = (A+B)x - 2A - 5B,$$

was zu $A = \frac{1}{3}$ und $B = -\frac{1}{3}$ führt. Also ist

$$\begin{aligned} \int_3^4 \frac{dx}{x^2 - 7x + 10} &= \int_3^4 \left(\frac{\frac{1}{3}}{x-5} - \frac{\frac{1}{3}}{x-2} \right) dx \\ &= \frac{1}{3} \ln |x-5| \Big|_{x=3}^{x=4} - \frac{1}{3} \ln |x-2| \Big|_{x=3}^{x=4} = -\frac{2}{3} \ln 2. \end{aligned}$$

d) Wir führen erneut eine Partialbruchzerlegung durch:

$$\frac{x-1}{x(x^2-2)} = \frac{x-1}{x(x-\sqrt{2})(x+\sqrt{2})} \stackrel{!}{=} \frac{A}{x} + \frac{B}{x-\sqrt{2}} + \frac{C}{x+\sqrt{2}}$$

$$\iff x - 1 = A(x-\sqrt{2})(x+\sqrt{2}) + Bx(x+\sqrt{2}) + Cx(x-\sqrt{2}).$$

Nun setzen wir die „kritischen“ x-Werte ein:

- $x = 0$:
$$-1 = -2A \Longrightarrow A = \frac{1}{2},$$
- $x = \sqrt{2}$:
$$\sqrt{2} - 1 = 4B \Longrightarrow B = \frac{\sqrt{2} - 1}{4},$$

Bitte wenden!

- $x = -\sqrt{2}$:

$$-\sqrt{2} - 1 = 4C \Longrightarrow C = -\frac{\sqrt{2}+1}{4}.$$

Einsetzen der Partialbruchzerlegung ergibt

$$\begin{aligned}
\int_2^3 \frac{x-1}{x(x^2-2)}\,dx &= \int_2^3 \left(\frac{1}{2x} + \frac{\sqrt{2}-1}{4}\frac{1}{x-\sqrt{2}} - \frac{\sqrt{2}+1}{4}\frac{1}{x+\sqrt{2}}\right)\,dx \\
&= \left(\frac{1}{2}\ln(x) + \frac{\sqrt{2}-1}{4}\ln(x-\sqrt{2}) - \frac{\sqrt{2}+1}{4}\ln(x+\sqrt{2})\right)\Bigg|_2^3 \\
&= \frac{1}{2}\ln\frac{3}{2} + \frac{\sqrt{2}-1}{4}\ln(\frac{3-\sqrt{2}}{2-\sqrt{2}}) - \frac{\sqrt{2}+1}{4}\ln(\frac{3+\sqrt{2}}{2+\sqrt{2}}).
\end{aligned}$$

Mit mehreren weiteren Rechenschritten erhält man ein leicht vereinfachtes Resultat:

$$\begin{aligned}
\int_2^3 \frac{x-1}{x(x^2-2)}\,dx &= \frac{1}{2}\ln\frac{3}{2} + \frac{\sqrt{2}-1}{4}\ln(\frac{(3-\sqrt{2})(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}) \\
&\quad -\frac{\sqrt{2}+1}{4}\ln(\frac{(3+\sqrt{2})(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}) \\
&= \frac{1}{2}\ln\frac{3}{2} + \frac{\sqrt{2}-1}{4}\ln(\frac{4+\sqrt{2}}{2}) - \frac{\sqrt{2}+1}{4}\ln(\frac{4-\sqrt{2}}{2}) \\
&= \frac{1}{2}(\ln 3 - \ln 2) + \frac{\sqrt{2}-1}{4}[\ln(4+\sqrt{2}) - \ln 2] \\
&\quad -\frac{\sqrt{2}+1}{4}[\ln(4-\sqrt{2}) - \ln 2] \\
&= \frac{1}{2}\ln 3 + \frac{1}{2\sqrt{2}}[\ln(4+\sqrt{2}) - \ln(-4+\sqrt{2})] \\
&\quad -\frac{1}{4}[\ln(4+\sqrt{2}) + \ln(4-\sqrt{2})] \\
&= \frac{1}{2}\ln 3 + \frac{1}{2\sqrt{2}}\ln\frac{4+\sqrt{2}}{4-\sqrt{2}} - \frac{1}{4}\ln[(4+\sqrt{2})(4-\sqrt{2})] \\
&= \frac{1}{2}\ln 3 + \frac{1}{2\sqrt{2}}\ln\frac{18+8\sqrt{2}}{14} - \frac{1}{4}\ln 14 \\
&= \frac{1}{2}\ln 3 - \frac{1}{4}\ln 14 + \frac{1}{2\sqrt{2}}\ln\frac{9+4\sqrt{2}}{7}.
\end{aligned}$$

Siehe nächstes Blatt!

e) Es ist

$$\int_3^4 \frac{dx}{x^2-2x+5} = \int\limits_3^4 \frac{dx}{(x-1)^2+4} = \int\limits_2^3 \frac{du}{u^2+4} = \int\limits_1^{\frac{3}{2}} \frac{\frac{1}{2}\,dv}{v^2+1}$$
$$= \frac{1}{2}\left(\arctan\frac{3}{2} - \arctan 1\right) = \frac{1}{2}\arctan\frac{3}{2} - \frac{\pi}{8},$$

wobei wir zuerst $u = x-1$ und dann $u = 2v$ substituiert haben.

Bitte wenden!

English version

1. **a)** By means of a partial decomposition, the integrand can be written as a sum, whose terms are easy to integrate:

$$\frac{1}{x^3-x} = \frac{1}{x(x+1)(x-1)} \overset{\text{(PBZ)}}{=} -\frac{1}{x} + \frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1} \tag{2}$$

Indeed, we do the Ansatzt

$$\frac{1}{x^3-x} = \frac{1}{x(x+1)(x-1)} = \frac{\omega_1}{x} + \frac{\omega_2}{x+1} + \frac{\omega_3}{x-1}$$

and get

$$\omega_1(x^2-1) + \omega_2 x(x-1) + \omega_3 x(x+1) = 1\,,$$

i.e.

$$\begin{cases} \omega_1+\omega_2+\omega_3 = 0\,, \\ -\omega_2+\omega_3 = 0\,, \\ -\omega_1 = 1 \end{cases}$$

The solutions of the system are the numbers

$$\omega_1 = -1\,,\ \ \omega_2 = \frac{1}{2}\,,\ \ \omega_3 = \frac{1}{2}\,,$$

and we get (2).

$$\begin{aligned}
\int_2^3 \frac{dx}{x^3-x} &= -\int_2^3 \frac{1}{x}\,dx + \frac{1}{2}\int_2^3 \frac{1}{x+1}\,dx + \frac{1}{2}\int_2^3 \frac{1}{x-1}\,dx \\
&= -\ln x\Big|_{x=2}^{x=3} + \frac{1}{2}\ln(x+1)\Big|_{x=2}^{x=3} + \frac{1}{2}\ln(x-1)\Big|_{x=2}^{x=3} \\
&= -\ln\frac{3}{2} + \frac{1}{2}\ln\frac{4}{3} + \frac{1}{2}\ln 2 \\
&= \ln\frac{2}{3} + \ln\frac{2}{\sqrt{3}} + \ln\sqrt{2} = \ln\left(\frac{2}{3}\cdot\frac{2}{\sqrt{3}}\cdot\sqrt{2}\right) \\
&= \ln\sqrt{\frac{32}{27}}
\end{aligned}$$

b) We use a partial decomposition: For the numerator we have

$$x^2-2x-63 = (x+7)(x-9)$$

and so we look for A, B so that

$$\frac{4x-2}{(x+7)(x-9)} = \frac{A}{x-9} + \frac{B}{x+7}.$$

Siehe nächstes Blatt!

We obtain

$$4x - 2 = A(x+7) + B(x-9).$$

We insert $x = -7$, respectively $x = 9$ and obtain

$$-30 = -16B \quad \Rightarrow B = 15/8$$

and

$$34 = 16 \cdot A \quad \Rightarrow A = 17/8$$

and so

$$\int \frac{4x-2}{x^2-2x-63} dx = \int \frac{17/8}{x-9} dx + \int \frac{15/8}{x+7} dx = \frac{17}{8} \log(|x-9|) + \frac{15}{8} \log(|x+7|) + c.$$

c) We use a partial decomposition with the Ansatz

$$\frac{2x+1}{(x+2)^2} = \frac{a}{x+2} + \frac{b}{(x+2)^2}\,.$$

This now gives

$$\frac{2x+1}{(x+2)^2} = \frac{ax+2a+b}{(x+2)^2}$$

and hence the system

$$\begin{cases} a = 2 \\ 2a + b = 1 \end{cases}$$

i.e. $a = 2$ and $b = -3$. We obtain thus

$$\begin{aligned} \int \frac{2x+1}{(x+2)^2}\, dx &= \int \frac{2}{x+2}\, dx - \int \frac{3}{(x+2)^2}\, dx \\ &= 2 \log|x+2| + \frac{3}{x+2} + C \end{aligned}$$

and we are done.

2. **a)** We factorise the denominator and get

$$(x^2-9)^2 = [(x-3)(x+3)]^2 = (x-3)^2(x+3)^2\,.$$

Because the zeros 3 and -3 have multiplicity 2, it follows that we must make the Ansatz

$$\frac{x^2}{(x^2-9)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2}$$

with A, B, C, D real numbers to be determined.

Hence, we get

$$\frac{A(x-3)(x+3)^2 + B(x+3)^2 + C(x-3)^2(x+3) + D(x-3)^2}{(x-3)^2(x+3)^2} = \frac{x^2}{(x-3)^2(x+3)^2}$$

and

$$(A+C)x^3+(3A+B-3C+D)x^2+(-9A+6B-9C-6D)x+(-27A+9B+27C+9D) = x^2\,,$$

which leads to the system

$$\begin{cases} A + C = 0 \\ 3A + B - 3C + D = 1 \\ -9A + 6B - 9C - 6D = 0 \\ -27A + 9B + 27C + 9D = 0\,. \end{cases}$$

The system admits the solutions

$$A = \frac{1}{12},\ B = \frac{1}{4},\ C = -\frac{1}{12},\ D = \frac{1}{4}$$

and so the decomposition is given by

$$\frac{x^2}{(x^2-9)^2} = \frac{1}{12(x-3)} + \frac{1}{4(x-3)^2} - \frac{1}{12(x+3)} + \frac{1}{4(x+3)^2}\,.$$

b) First of all, we decompose the given rational function by division into the sum of a polynomial $p(x)$ and a rational rest $r(x)$, so that the degree of the numerator of $r(x)$ is smaller than the degree of the denominator. We get

$$(x^{10} - x^7 + 3x) : (x^3 - 1) = x^7 + \frac{3x}{x^3-1}\,.$$

We factorise the denominator of $r(x) = \frac{3x}{x^3-1}$ and get

$$r(x) = \frac{3x}{(x-1)(x^2+x+1)}\,.$$

In this case, the Ansatz is

$$\frac{3x}{x^3-1} = \frac{a}{x-1} + \frac{bx+c}{x^2+x+1}\,.$$

Thus, we get

$$\frac{3x}{x^3-1} = \frac{ax^2+ax+a+bx^2-bx+cx-c}{x^3-1}$$

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and the system

$$\begin{cases} a + b = 0 \\ a - b + c = 3 \\ a - c = 0\,, \end{cases}$$

whose solutions are the numbers

$$a = 1,\ b = -1,\ c = 1\,.$$

The decomposition is given by

$$\frac{x^{10} - x^7 + 3x}{x^3 - 1} = x^7 + \frac{1}{x-1} + \frac{1-x}{x^2+x+1}\,.$$

3. a) The substitution $t = \sin x$ gives

$$\int \frac{\cos^3 x}{\sin^4 x}\,dx = \int \frac{(1-\sin^2 x)\cos x}{\sin^4 x}\,dx = \int \frac{1-t^2}{t^4}\,dt = \int (\frac{1}{t^4} - \frac{1}{t^2})\,dt$$
$$= -\frac{1}{3t^3} + \frac{1}{t} + K = -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + K.$$

b) The substitution $x = 4\sinh z$ gives $dx = 4\cosh z\,dz$ and

$$\int \sqrt{x^2+16}\,dx = \int \sqrt{16\sinh^2 z + 16}\cdot 4\cosh z\,dz = 16\int \cosh^2 z\,dz$$
$$= 8\int (\cosh 2z + 1)\,dz = 4\sinh 2z + 8z + C$$
$$= 8\sinh z\cosh z + 8z + C$$
$$= 2x\cosh\big(\operatorname{Arsinh}\frac{x}{4}\big) + 8\operatorname{Arsinh}\tfrac{x}{4} + C$$
$$= 2x\sqrt{\left(\frac{x}{4}\right)^2 + 1} + 8\operatorname{Arsinh}\frac{x}{4} + C$$
$$= \frac{1}{2}\,x\sqrt{x^2+16} + 8\operatorname{Arsinh}\frac{x}{4} + C.$$

Here we have used the relation

$$\cosh^2 z = \frac{1}{2}\left(\cosh(2z) + 1\right)\,.$$

Remark: Alternatively, one can compute the above integral $I := \int \cosh^2 z\,dz$ also with the help of a partial integration:

$$I = \int \cosh^2 z\,dz = \sinh z\cosh z - \int \sinh^2 z\,dz$$
$$= \sinh z\cosh z - \int (\cosh^2 z - 1)\,dz = \sinh z\cosh z - \int \cosh^2 z\,dz + z$$
$$= \sinh z\cosh z + z - I,$$

hence

$$I = \frac{1}{2}\left(\sinh z \cosh z + z\right).$$

c) We deal with this integral with a partial decomposition:

$$\frac{1}{x^2 - 7x + 10} = \frac{1}{(x-5)(x-2)} \stackrel{!}{=} \frac{A}{x-5} + \frac{B}{x-2}.$$

Multiplication with $(x-5)(x-2)$ gives

$$1 = (x-2)A + (x-5)B = (A+B)x - 2A - 5B,$$

which leads to $A = \frac{1}{3}$ and $B = -\frac{1}{3}$. Hence, we have

$$\begin{aligned}\int_3^4 \frac{dx}{x^2 - 7x + 10} &= \int_3^4 \left(\frac{\frac{1}{3}}{x-5} - \frac{\frac{1}{3}}{x-2}\right) dx \\ &= \frac{1}{3}\ln|x-5|\Big|_{x=3}^{x=4} - \frac{1}{3}\ln|x-2|\Big|_{x=3}^{x=4} = -\frac{2}{3}\ln 2.\end{aligned}$$

d) We perform again a partial decomposition:

$$\frac{x-1}{x(x^2-2)} = \frac{x-1}{x(x-\sqrt{2})(x+\sqrt{2})} \stackrel{!}{=} \frac{A}{x} + \frac{B}{x-\sqrt{2}} + \frac{C}{x+\sqrt{2}}$$

$$\Longleftrightarrow x - 1 = A(x-\sqrt{2})(x+\sqrt{2}) + Bx(x+\sqrt{2}) + Cx(x-\sqrt{2}).$$

We now insert the „critical" x-values:

- $x = 0$:
$$-1 = -2A \Longrightarrow A = \frac{1}{2},$$
- $x = \sqrt{2}$:
$$\sqrt{2} - 1 = 4B \Longrightarrow B = \frac{\sqrt{2}-1}{4},$$
- $x = -\sqrt{2}$:
$$-\sqrt{2} - 1 = 4C \Longrightarrow C = -\frac{\sqrt{2}+1}{4}.$$

Therefore, we obtain

$$\begin{aligned}\int_2^3 \frac{x-1}{x(x^2-2)}\,dx &= \int_2^3 \left(\frac{1}{2x} + \frac{\sqrt{2}-1}{4}\frac{1}{x-\sqrt{2}} - \frac{\sqrt{2}+1}{4}\frac{1}{x+\sqrt{2}}\right) dx \\ &= \left(\frac{1}{2}\ln(x) + \frac{\sqrt{2}-1}{4}\ln(x-\sqrt{2}) - \frac{\sqrt{2}+1}{4}\ln(x+\sqrt{2})\right)\Bigg|_2^3 \\ &= \frac{1}{2}\ln\frac{3}{2} + \frac{\sqrt{2}-1}{4}\ln(\frac{3-\sqrt{2}}{2-\sqrt{2}}) - \frac{\sqrt{2}+1}{4}\ln(\frac{3+\sqrt{2}}{2+\sqrt{2}}).\end{aligned}$$

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With some more computations, one can obtain a simpler result:

$$\begin{aligned}
\int_2^3 \frac{x-1}{x(x^2-2)}\,dx &= \frac{1}{2}\ln\frac{3}{2} + \frac{\sqrt{2}-1}{4}\ln(\frac{(3-\sqrt{2})(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}) \\
&\quad -\frac{\sqrt{2}+1}{4}\ln(\frac{(3+\sqrt{2})(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}) \\
&= \frac{1}{2}\ln\frac{3}{2} + \frac{\sqrt{2}-1}{4}\ln(\frac{4+\sqrt{2}}{2}) - \frac{\sqrt{2}+1}{4}\ln(\frac{4-\sqrt{2}}{2}) \\
&= \frac{1}{2}(\ln 3 - \ln 2) + \frac{\sqrt{2}-1}{4}[\ln(4+\sqrt{2}) - \ln 2] \\
&\quad -\frac{\sqrt{2}+1}{4}[\ln(4-\sqrt{2}) - \ln 2] \\
&= \frac{1}{2}\ln 3 + \frac{1}{2\sqrt{2}}[\ln(4+\sqrt{2}) - \ln(-4+\sqrt{2})] \\
&\quad -\frac{1}{4}[\ln(4+\sqrt{2}) + \ln(4-\sqrt{2})] \\
&= \frac{1}{2}\ln 3 + \frac{1}{2\sqrt{2}}\ln\frac{4+\sqrt{2}}{4-\sqrt{2}} - \frac{1}{4}\ln[(4+\sqrt{2})(4-\sqrt{2})] \\
&= \frac{1}{2}\ln 3 + \frac{1}{2\sqrt{2}}\ln\frac{18+8\sqrt{2}}{14} - \frac{1}{4}\ln 14 \\
&= \frac{1}{2}\ln 3 - \frac{1}{4}\ln 14 + \frac{1}{2\sqrt{2}}\ln\frac{9+4\sqrt{2}}{7}.
\end{aligned}$$

e) We have

$$\begin{aligned}
\int_3^4 \frac{dx}{x^2-2x+5} &= \int_3^4 \frac{dx}{(x-1)^2+4} = \int_2^3 \frac{du}{u^2+4} = \int_1^{\frac{3}{2}} \frac{\frac{1}{2}\,dv}{v^2+1} \\
&= \frac{1}{2}\left(\arctan\frac{3}{2} - \arctan 1\right) = \frac{1}{2}\arctan\frac{3}{2} - \frac{\pi}{8},
\end{aligned}$$

whereas first we have substituted $u = x-1$ and then $u = 2v$.