Musterlösung 8

1. **a)** Der Ausdruck $\log(\sin x)$ ist für $x \in (0, \pi)$ wohldefiniert, da dann $\sin(x) > 0$ gilt. Anwendung der Kettenregel ergibt

$$\frac{d}{dx}(\log(\sin(x))) = \frac{1}{\sin(x)} \sin'(x) = \frac{\cos(x)}{\sin(x)}.$$

b) Wir schreiben $a^x = e^{x \log(a)}$ und erhalten

$$\frac{d}{dx}(a^x) = \log(a) e^{x \log(a)} = \log(a) a^x.$$

c) Es gilt $x^x = e^{x \log x}$. Unter Anwendung der Ketten- und anschliessend der Produktregel erhalten wir

$$\frac{d}{dx}(x^x) = \left[\frac{d}{dx}(x \log x)\right] e^{x \log x} = \left(\frac{x}{x} + \log x\right) e^{x \log x} = (1 + \log x) x^x.$$

d) Es gilt

$$\begin{aligned} \frac{d}{dx}(9x^7 + 3x^{-5} - 3x^{-11}) &= 7 \cdot 9x^6 + (-5)3x^{-6} - (-11)3x^{-12} \\ &= 63x^6 - 15x^{-6} + 33x^{-12}. \end{aligned}$$

e) Wir verwenden zuerst die Kettenregel und dann die Quotientenregel:

$$\begin{aligned} \frac{d}{dx}\sqrt{\frac{x^2 - 3x + 2}{x^2 - 7x + 12}} &= \frac{1}{2\sqrt{\frac{x^2-3x+2}{x^2-7x+12}}} \cdot \frac{d}{dx}\left(\frac{x^2 - 3x + 2}{x^2 - 7x + 12}\right) \\ &= \frac{1}{2\sqrt{\frac{x^2-3x+2}{x^2-7x+12}}} \cdot \frac{(2x-3)(x^2 - 7x + 12) - (x^2 - 3x + 2)(2x - 7)}{(x^2 - 7x + 12)^2} \\ &= \frac{(x^2 - 7x + 12)^{\frac{1}{2}}}{2(x^2 - 3x + 2)^{\frac{1}{2}}} \cdot \frac{(-4x^2 + 20x - 22)}{(x^2 - 7x + 12)^2} \\ &= \frac{-2x^2 + 10x - 11}{(x^2 - 3x + 2)^{\frac{1}{2}} (x^2 - 7x + 12)^{\frac{3}{2}}}. \end{aligned}$$

Bitte wenden!

f) Mit den Beziehungen

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x \quad \text{und} \quad \frac{d}{dx}(\log x) = \frac{1}{x}$$

und der Kettenregel folgt

$$\frac{d}{dx}(\log(\cosh x)) = \frac{\frac{d}{dx}(\cosh x)}{\cosh x} = \frac{\sinh x}{\cosh x} = \tanh x.$$

g) Mit der Kettenregel folgt:

$$\frac{d}{dx}(\log(\log(\log x))) = \frac{1}{\log(\log x)} \cdot \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x \cdot \log x \cdot \log(\log x)}.$$

h) Es gilt $a^b = e^{b \log a}$. Daher ist $3^x = e^{x \log 3}$ und es folgt

$$\frac{d}{dx}(3^x) = \frac{d}{dx}(e^{x \log 3}) = (\log 3)e^{x \log 3} = (\log 3)3^x.$$

Mit der Produktregel erhalten wir:

$$\begin{aligned} \frac{d}{dx}3^x x^3 &= 3^x \cdot 3x^2 + (\log 3)3^x x^3 \\ &= 3^x x^2(3 + x \log 3). \end{aligned}$$

i) Die Funktion $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ ist die Umkehrfunktion von $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$. Für $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ ist

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{d}{dx}\frac{\sin x}{\cos x} \\ &= \frac{\left[\frac{d}{dx}(\sin x)\right] \cdot \cos x - \sin x \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= 1 + \tan^2 x. \end{aligned}$$

Mit der Regel

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

erhalten wir also

$$\begin{aligned} \frac{d}{dx}(\arctan x) &= \frac{1}{\tan'(\arctan x)} \\ &= \frac{1}{1 + \tan^2(\arctan x)} \\ &= \frac{1}{1 + x^2} \end{aligned}$$

Siehe nächstes Blatt!

und somit gilt

$$
\begin{aligned}
\frac{d}{dx}\arctan(x-\sqrt{x^2+1}) &= \frac{1}{1+(x-\sqrt{x^2+1})^2}\left(1-\frac{2x}{2\sqrt{x^2+1}}\right) \\
&= \frac{1-\frac{x}{\sqrt{x^2+1}}}{1+(x-\sqrt{x^2+1})^2} \\
&= \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}(1+(x-\sqrt{x^2+1})^2)} \\
&= \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}(1+x^2-2x\sqrt{x^2+1}+(x^2+1))} \\
&= \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}(-2x\sqrt{x^2+1}+2+2x^2)} \\
&= \frac{x^2+1-x\sqrt{x^2+1}}{2(x^2+1)(x^2+1-x\sqrt{x^2+1})} \\
&= \frac{1}{2(x^2+1)}.
\end{aligned}
$$

2. Auf ganz $\mathbb{R}\setminus\{0\}$ ist k stetig. An der Stelle 0 gilt $k(0) = 1 = \lim_{x\to 0^-} k(x)$ und $\lim_{x\to 0^+} k(x) = \alpha\cdot 0 + t\cdot 0 + r = r$. Daher ist k stetig an der Stelle 0 für $\alpha, t \in \mathbb{R}$ und $r = 1$.

Ausserdem gilt $k'(x) = 100x^{99}$ für $x < 0$ und der linksseitige Limes $\lim_{x\to 0^-} k'(x) = 0$ existiert. Für $x > 0$ gilt

$$k'(x) = \frac{\alpha}{100}x^{1/100-1} + te^{-1/x}x^{-2} = \frac{\alpha}{100}x^{-99/100} + t\frac{e^{-1/x}}{x^2}\,.$$

Wir wollen nun $\lim_{x\to 0^+} k'(x)$ berechnen. Es gilt $\lim_{x\to 0^+} t\frac{e^{-1/x}}{x^2} = 0$ für alle $t\in\mathbb{R}$. Ausserdem sehen wir, dass

$$\lim_{x\to 0^+}\frac{\alpha}{100}x^{-99/100} = \begin{cases} \infty & \text{falls } \alpha > 0\,, \\ -\infty & \text{falls } \alpha < 0\,, \\ 0 & \text{falls } \alpha = 0\,, \end{cases}$$

ist. Dann gilt für $\alpha = 0$ und $t\in\mathbb{R}$, dass $\lim_{x\to 0^+} k'(x) = 0 = \lim_{x\to 0^-} k'(x)$ ist. Wir schliessen, dass k differenzierbar auf ganz $\mathbb{R}$ für $\alpha = 0, t\in\mathbb{R}$ und $r = 1$ ist.

3. Weil $f(x) \geq 0$ für alle $x\in\mathbb{R}$ und $f(0) = 0$ ist, erhalten wir sofort, dass $x = 0$ ein globale Minimalstelle ist.

Bitte wenden!

Weil für alle $x \in \mathbb{R}$ $x^2 - 3x + 3 > 0$ ist, sehen wir, dass $x^3 - 3x^2 + 3x = x(x^2 - 3x + 3) \geq 0$ genau dann, wenn $x \geq 0$ ist. Darum ist $x = 0$ die globale Minimalstelle. Ferner können wir schreiben

$$f(x) = \begin{cases} x^3 - 3x^2 + 3x & \text{falls } x \geq 0\,, \\ -x^3 + 3x^2 - 3x & \text{falls } x < 0\,. \end{cases}$$

Deshalb folgt

$$f'(x) = \begin{cases} 3x^2 - 6x + 3 & \text{falls } x > 0\,, \\ -3x^2 + 6x - 3 & \text{falls } x < 0\,. \end{cases}$$

Da $3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2 \geq 0$ für alle $x \in \mathbb{R}$ ist, erhalten wir, dass f monoton wachsend auf $(0, 3]$ und monoton fallend auf $[-2, 0)$ ist.

Daher sind $x = -2$ und $x = 3$ relative Maximalstellen. Ferner gilt $f(-2) = 26 > 9 = f(3)$ und somit ist $x = -2$ die globale Maximalstelle.

Siehe nächstes Blatt!

English version

1. **a)** The expression $\log(\sin x)$ is meaningful for $x \in (0, \pi)$, because $\sin(x) > 0$. We apply the chain rule

$$\frac{d}{dx}(\log(\sin(x))) = \frac{1}{\sin(x)} \sin'(x) = \frac{\cos(x)}{\sin(x)}.$$

b) We write $a^x = e^{x \log(a)}$ and get

$$\frac{d}{dx}(a^x) = \log(a) e^{x \log(a)} = \log(a) a^x.$$

c) We have $x^x = e^{x \log x}$. By means of the chain- and product rule we obtain

$$\frac{d}{dx}(x^x) = \left[\frac{d}{dx}(x \log x)\right] e^{x \log x} = \left(\frac{x}{x} + \log x\right) e^{x \log x} = (1 + \log x) x^x.$$

d) We have

$$\begin{aligned} \frac{d}{dx}(9x^7 + 3x^{-5} - 3x^{-11}) &= 7 \cdot 9x^6 + (-5)3x^{-6} - (-11)3x^{-12} \\ &= 63x^6 - 15x^{-6} + 33x^{-12}. \end{aligned}$$

e) We apply first the chain rule and afterward the quotient rule:

$$\begin{aligned} \frac{d}{dx}\sqrt{\frac{x^2-3x+2}{x^2-7x+12}} &= \frac{1}{2\sqrt{\frac{x^2-3x+2}{x^2-7x+12}}} \cdot \frac{d}{dx}\left(\frac{x^2-3x+2}{x^2-7x+12}\right) \\ &= \frac{1}{2\sqrt{\frac{x^2-3x+2}{x^2-7x+12}}} \cdot \frac{(2x-3)(x^2-7x+12)-(x^2-3x+2)(2x-7)}{(x^2-7x+12)^2} \\ &= \frac{(x^2-7x+12)^{\frac{1}{2}}}{2(x^2-3x+2)^{\frac{1}{2}}} \cdot \frac{(-4x^2+20x-22)}{(x^2-7x+12)^2} \\ &= \frac{-2x^2+10x-11}{(x^2-3x+2)^{\frac{1}{2}}(x^2-7x+12)^{\frac{3}{2}}}. \end{aligned}$$

f) With the relation

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x \quad \text{and} \quad \frac{d}{dx}(\log x) = \frac{1}{x}$$

and the chain rule it follows

$$\frac{d}{dx}(\log(\cosh x)) = \frac{\frac{d}{dx}(\cosh x)}{\cosh x} = \frac{\sinh x}{\cosh x} = \tanh x.$$

Bitte wenden!

g) With the chain rule one obtains:

$$\frac{d}{dx}(\log(\log(\log x))) = \frac{1}{\log(\log x)} \cdot \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x \cdot \log x \cdot \log(\log x)}.$$

h) We have $a^b = e^{b \log a}$. Hence, we get $3^x = e^{x \log 3}$ and it follows

$$\frac{d}{dx}(3^x) = \frac{d}{dx}(e^{x \log 3}) = (\log 3)e^{x \log 3} = (\log 3)3^x.$$

With the product rule we obtain:

$$\begin{aligned} \frac{d}{dx}3^x x^3 &= 3^x \cdot 3x^2 + (\log 3)3^x x^3 \\ &= 3^x x^2(3 + x \log 3). \end{aligned}$$

i) The function $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is the inverse function of $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$. For $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we have

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{d}{dx}\frac{\sin x}{\cos x} \\ &= \frac{\left[\frac{d}{dx}(\sin x)\right] \cdot \cos x - \sin x \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= 1 + \tan^2 x. \end{aligned}$$

With the formula

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

we obtain

$$\begin{aligned} \frac{d}{dx}(\arctan x) &= \frac{1}{\tan'(\arctan x)} \\ &= \frac{1}{1 + \tan^2(\arctan x)} \\ &= \frac{1}{1 + x^2} \end{aligned}$$

Siehe nächstes Blatt!

and thus

$$
\begin{aligned}
\frac{d}{dx}\arctan(x-\sqrt{x^2+1}) &= \frac{1}{1+(x-\sqrt{x^2+1})^2}\left(1-\frac{2x}{2\sqrt{x^2+1}}\right)\\
&= \frac{1-\frac{x}{\sqrt{x^2+1}}}{1+(x-\sqrt{x^2+1})^2}\\
&= \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}(1+(x-\sqrt{x^2+1})^2)}\\
&= \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}(1+x^2-2x\sqrt{x^2+1}+(x^2+1))}\\
&= \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}(-2x\sqrt{x^2+1}+2+2x^2)}\\
&= \frac{x^2+1-x\sqrt{x^2+1}}{2(x^2+1)(x^2+1-x\sqrt{x^2+1})}\\
&= \frac{1}{2(x^2+1)}.
\end{aligned}
$$

2. On the whole $\mathbb{R}\backslash\{0\}$ k is continuous. At the point 0 we have $k(0) = 1 = \lim_{x\to 0^-} k(x)$ and $\lim_{x\to 0^+} k(x) = \alpha\cdot 0 + t\cdot 0 + r = r$. Hence k is continuous at 0 for $\alpha, t \in \mathbb{R}$ and $r = 1$.

Furthermore, we have $k'(x) = 100x^{99}$ for $x < 0$ and the left limit $\lim_{x\to 0^-} k'(x) = 0$ exists. For $x > 0$ we have

$$k'(x) = \frac{\alpha}{100}x^{1/100-1} + te^{-1/x}x^{-2} = \frac{\alpha}{100}x^{-99/100} + t\frac{e^{-1/x}}{x^2}\,.$$

We want now to compute $\lim_{x\to 0^+} k'(x)$. We have $\lim_{x\to 0^+} t\frac{e^{-1/x}}{x^2} = 0$ for all $t \in \mathbb{R}$. Moreover, we see that

$$\lim_{x\to 0^+}\frac{\alpha}{100}x^{-99/100} = \begin{cases} \infty & \text{if } \alpha > 0\,,\\ -\infty & \text{if } \alpha < 0\,,\\ 0 & \text{if } \alpha = 0\,.\end{cases}$$

Then for $\alpha = 0$ and $t \in \mathbb{R}$ one has $\lim_{x\to 0^+} k'(x) = 0 = \lim_{x\to 0^-} k'(x)$. We conclude that k is differentiable on the whole $\mathbb{R}$ for $\alpha = 0, t \in \mathbb{R}$ and $r = 1$.

3. Since $f(x) \geq 0$ for all $x \in \mathbb{R}$ and $f(0) = 0$, we get that $x = 0$ is a global point of minimum.

Since for all $x \in \mathbb{R}$ $x^2 - 3x + 3 > 0$, we see that $x^3 - 3x^2 + 3x = x(x^2 - 3x + 3) \geq 0$ iff $x \geq 0$. Thus, $x = 0$ is the global point of minimum. Moreover, we can write

$$f(x) = \begin{cases} x^3 - 3x^2 + 3x & \text{if } x \geq 0\,, \\ -x^3 + 3x^2 - 3x & \text{if } x < 0\,. \end{cases}$$

It follows

$$f'(x) = \begin{cases} 3x^2 - 6x + 3 & \text{if } x > 0\,, \\ -3x^2 + 6x - 3 & \text{if } x < 0\,. \end{cases}$$

Since $3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2 \geq 0$ for all $x \in \mathbb{R}$, we obtain that f is monoton increasing on $(0, 3]$ and monoton decreasing on $[-2, 0)$.

Hence, $x = -2$ and $x = 3$ are relative points of minimum. Moreover, one has $f(-2) = 26 > 9 = f(3)$ and thus $x = -2$ is the global point of maximum.