Musterlösung Schnellübung 5

1. Betrachte die Funktion $f(x) = e^{x^2}$. Die n-te Taylor-Entwicklung von f um den Punkt $x = 0$ lautet:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x).$$

Dabei ist das Restglied gleich

$$R_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!} x^{n+1}$$

für ein $\tau \in (0, 1/4)$. Wir müssen n so wählen, dass gilt $|R_n(1/4)| < 5 \cdot 10^{-3}$.

$$\begin{aligned} f(x) &= e^{x^2} & f(0) &= 1 \\ f'(x) &= 2xe^{x^2} & f'(0) &= 0 \\ f''(x) &= (4x^2+2)e^{x^2} & f''(0) &= 2 \\ f'''(x) &= (8x^3+12x)e^{x^2} & f'''(0) &= 0 \\ f^{(4)}(x) &= (16x^4+48x^2+12)e^{x^2} & f^{(4)}(0) &= 12 \end{aligned}$$

Mit der Abschätzung $e^{1/16} \leq \sqrt[16]{4} = \sqrt[8]{2} \leq \sqrt{2} \leq 3/2$ folgt

$$|R_3(1/4)| \leq \frac{(16/4^4 + 48/4^2 + 12) \cdot (3/2)}{4!} (1/4)^4 \leq \frac{1}{4^4} \leq 5 \cdot 10^{-3}.$$

Für $n = 3$ lautet das Taylorpolynom

$$f(0) + f'(0) \cdot x + \frac{f''(0)}{2} \cdot x^2 = 1 + x^2.$$

Dessen Wert an der Stelle $x = 1/4$ liefert die gesuchte Näherung

$$1 + \frac{1}{4^2} = 1.0625$$

(Der wahre Wert ist $f(1/4) = 1.0644944\ldots$)

2. Allgemein:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_n(x)$$

$R_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}(x - x_0)^{n+1}$,wobei τ zwischen x und x_0

Taylorpolynom für $f(x) = \log(1 + x)$ um $x_0 = 0$:

Bitte wenden!

$$
\begin{array}{ll}
f(x) = \log(1+x) & f(0) = \log(1) = 0 \\
f'(x) = \frac{1}{1+x} = (1+x)^{-1} & f'(0) = 1 \\
f''(x) = -\frac{1}{(1+x)^2} = -(1+x)^{-2} & f''(0) = -1 \\
f'''(x) = 2\frac{1}{(1+x)^3} & f'''(0) = 2 \\
f^{(4)}(x) = -3!\frac{1}{(1+x)^4} & f^{(4)}(0) = -3! \\
 & \\
f^{(k)}(x) = (-1)^{k-1}\frac{(k-1)!}{(1+x)^k} & f^{(k)}(0) = (-1)^{(k-1)}(k-1)!
\end{array}
$$

$$
\begin{aligned}
\log(1+x) &= \sum_{k=0}^{n} (-1)^{(k-1)} \frac{(k-1)!}{k!} x^k + \frac{f^{(n+1)}(\tau)}{(n+1)!} x^{(n+1)} \\
&= \sum_{k=1}^{n} \frac{(-1)^{(k-1)}}{k} x^k + \frac{(-1)^n n!}{(n+1)!(1+\tau)^{n+1}} x^{(n+1)}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \log(1+x) - \sum_{k=1}^{2n-1} \frac{(-1)^{(k-1)}}{k} x^k &= (-1)^{2n-1} \frac{(2n-1)!}{(2n)!(1+\tau)^{2n}} x^{2n} \\
&= (-1)^{2n-1} \frac{1}{2n(1+\tau)^{2n}} x^{2n} < 0
\end{aligned}
$$

$$
\Rightarrow \log(1+x) < \sum_{k=1}^{2n-1} \frac{(-1)^{(k-1)}}{k} x^k \qquad \square
$$

3. **a)** Das charakteristische Polynom ist

$$\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1)$$

mit den beiden verschiedenen Nullstellen $\lambda = -2$ und $\lambda = 1$. Die allgemeine Lösung ist deshalb

$$y(x) = Ae^{-2x} + Be^{x}.$$

b) Das charakteristische Polynom ist

$$\lambda^2 + 6\lambda + 13$$

mit den beiden verschiedenen Nullstellen $\lambda = -3 + 2i$ und $\lambda = -3 - 2i$. Die allgemeine Lösung ist deshalb

$$y(x) = Ae^{(-3+2i)x} + Be^{(-3-2i)x}$$

oder

$$y(x) = \tilde{A}e^{-3x}\cos(2x) + \tilde{B}e^{-3x}\sin 2x.$$

Siehe nächstes Blatt!

c) Das charakteristische Polynom

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

hat eine doppelte Nullstelle bei $\lambda = 1$. Die allgemeine Lösung ist deshalb

$$y(x) = Ae^x + Bxe^x.$$

d) Das charakteristische Polynom

$$\lambda^4 - 4\lambda^2 = \lambda^2(\lambda + 2)(\lambda - 2)$$

hat eine doppelte Nullstelle bei $\lambda = 0$ und einfache Nullstellen bei $\lambda = -2$ und $\lambda = 2$. Die allgemeine Lösung ist deshalb

$$y(x) = A + Bx + Ce^{-2x} + De^{2x}.$$

Bitte wenden!

English version

1. Consider the function $f(x) = e^{x^2}$. The n-th Taylor-expansion of f about the point $x = 0$ is:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x).$$

Here, the rest term is equal to

$$R_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!} x^{n+1}$$

for a $\tau \in (0, 1/4)$. We must choose n such that holds true $|R_n(1/4)| < 5 \cdot 10^{-3}$.

$$\begin{array}{rclrcl} f(x) & = & e^{x^2} & f(0) & = & 1 \\ f'(x) & = & 2xe^{x^2} & f'(0) & = & 0 \\ f''(x) & = & (4x^2+2)e^{x^2} & f''(0) & = & 2 \\ f'''(x) & = & (8x^3+12x)e^{x^2} & f'''(0) & = & 0 \\ f^{(4)}(x) & = & (16x^4+48x^2+12)e^{x^2} & f^{(4)}(0) & = & 12 \end{array}$$

By the estimate $e^{1/16} \leq \sqrt[16]{4} = \sqrt[8]{2} \leq \sqrt{2} \leq 3/2$ follows

$$|R_3(1/4)| \leq \frac{(16/4^4 + 48/4^2 + 12) \cdot (3/2)}{4!} (1/4)^4 \leq \frac{1}{4^4} \leq 5 \cdot 10^{-3}.$$

For $n = 3$ the Taylor polynomial is

$$f(0) + f'(0) \cdot x + \frac{f''(0)}{2} \cdot x^2 = 1 + x^2,$$

whose value at the point $x = 1/4$ gives the wanted approximation

$$1 + \frac{1}{4^2} = 1.0625$$

(The true value is $f(1/4) = 1.0644944\ldots$)

2. In general:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_n(x)$$

$R_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}(x - x_0)^{n+1}$,whereas τ is between x and x_0

The Taylor polynomial for $f(x) = \log(1 + x)$ about $x_0 = 0:$

Siehe nächstes Blatt!

$$
\begin{array}{ll}
f(x) = \log(1+x) & f(0) = \log(1) = 0 \\
f'(x) = \frac{1}{1+x} = (1+x)^{-1} & f'(0) = 1 \\
f''(x) = -\frac{1}{(1+x)^2} = -(1+x)^{-2} & f''(0) = -1 \\
f'''(x) = 2\frac{1}{(1+x)^3} & f'''(0) = 2 \\
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 & \\
f^{(k)}(x) = (-1)^{k-1}\frac{(k-1)!}{(1+x)^k} & f^{(k)}(0) = (-1)^{(k-1)}(k-1)!
\end{array}
$$

$$
\begin{aligned}
\log(1+x) &= \sum_{k=0}^{n}(-1)^{(k-1)}\frac{(k-1)!}{k!}x^k + \frac{f^{(n+1)}(\tau)}{(n+1)!}x^{(n+1)} \\
&= \sum_{k=1}^{n}\frac{(-1)^{(k-1)}}{k}x^k + \frac{(-1)^n n!}{(n+1)!(1+\tau)^{n+1}}x^{(n+1)}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \log(1+x) - \sum_{k=1}^{2n-1}\frac{(-1)^{(k-1)}}{k}x^k &= (-1)^{2n-1}\frac{(2n-1)!}{(2n)!(1+\tau)^{2n}}x^{2n} \\
&= (-1)^{2n-1}\frac{1}{2n(1+\tau)^{2n}}x^{2n} < 0 \\
\Rightarrow \log(1+x) < \sum_{k=1}^{2n-1}\frac{(-1)^{(k-1)}}{k}x^k & \qquad \square
\end{aligned}
$$

3. **a)** The characteristic polynomial is

$$\lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1)$$

which admits the two different roots $\lambda = -2$ and $\lambda = 1$. The general solution is therefore

$$y(x) = Ae^{-2x} + Be^{x}.$$

b) The characteristic polynomial is

$$\lambda^2 + 6\lambda + 13$$

which admits the two different roots $\lambda = -3 + 2i$ and $\lambda = -3 - 2i$. The general solution is therefore

$$y(x) = Ae^{(-3+2i)x} + Be^{(-3-2i)x}$$

or

$$y(x) = \tilde{A}e^{-3x}\cos(2x) + \tilde{B}e^{-3x}\sin 2x.$$

c) The characteristic polynomial is

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

which has a root of multiplicity two at $\lambda = 1$. The general solution is therefore

$$y(x) = Ae^x + Bxe^x.$$

d) The characteristic polynomial is

$$\lambda^4 - 4\lambda^2 = \lambda^2(\lambda + 2)(\lambda - 2)$$

which has a root of multiplicity two at $\lambda = 0$ and two other simple roots at $\lambda = -2$ and $\lambda = 2$. The general solution is therefore

$$y(x) = A + Bx + Ce^{-2x} + De^{2x}.$$