The knitting algorithm

Finite root systems

Let V be an Euclidean space with inner product $(\cdot,\cdot)$.

Definition 1. *A subset R of V is called a* ***finite root system****, if it satisfies for $\alpha, \beta \in R$:*

(R1) R *is finite,* $\langle R \rangle = V$, $0 \notin R$

(R2) $\lambda\alpha \in R, \lambda \in \mathbb{R} \Rightarrow \lambda = \pm 1$

(R3) $\forall \gamma \in R : \sigma_\alpha(\gamma) \in R$*, where* $\sigma_\alpha(\gamma) := \gamma - \frac{2(\gamma,\alpha)}{(\alpha,\alpha)}$.

(R4) $\langle \alpha, \beta \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$

When we talk of a root system in this paper, we actually mean a finite root system. The ***rank*** *of a root system R is the dimension of V.*

Definition 2. *R is called* ***irreducible*** *if there is no representation $R = R_1 \cup R_2$ with two proper subsets R_1 and R_2 and $\forall x \in R_1\ \forall y \in R_2 : (x,y) = 0$.*

Definition 3. *A subset Δ of a root system R which is a basis for V is called* ***basis*** *of R, if for all $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ the coefficients k_α are integers and all less or equal to 0 or all greater or equal to 0. The elements in the basis Δ are called* ***simple roots****. Because Δ is a basis of V, the representation $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ is well-defined. If $\sum_{\alpha\in\Delta} k_\alpha > 0$ ($\sum_{\alpha\in\Delta} k_\alpha < 0$), then the root is called* ***positive (negative) root****.*

Proposition 1. *Every finite root system has a basis.*

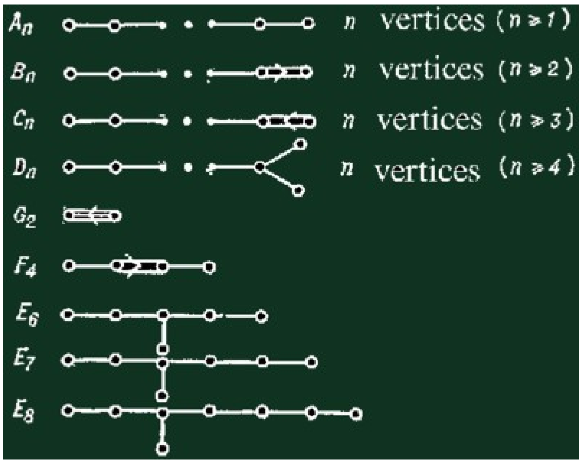
Dynkin-diagram

Let R be a finite root system.

Definition 4. *Let $\alpha_1, \ldots, \alpha_l$ be simple roots of R of rank l. Then the $l \times l$-matrix $(\langle \alpha_i, \alpha_j \rangle)_{ij}$ is called the* ***Cartan-matrix*** *of R.*

Proposition 2. *The Cartan-matrix of R is well-defined, i.e. independent of the choice of a basis of R.*

Definition 5. *Let $\alpha_1, \ldots, \alpha_l$ be the simple roots of R. We can construct a graph as follows: Take l vertices. Between vertex i and j we draw $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges (well-defined, because Cartan-matrix is independent of basis). There is an arrow on the edges from the larger root to the shorter one. This graph is called* ***Dynkin-diagram*** *of R.*



Remark 1. *R is irreducible if and only if the Dynkin-diagram of R is connected.*

Theorem 1. *Let R be an irreducible root system. Then its Dynkin-diagram is one of the following (and all diagrams can be realized as diagrams of irreducible root systems):*

Remark 2. *These diagrams also classify the finite dimensional simple complex Lie-Algebras.*

Definition 6. *Let Λ be a commutative ring. An **algebra** over Λ is a Λ-module A together with a bilinear map $A \times A \to A$.*

Definition 7. *A **cluster algebra** is a commutative $\mathbb{Q}$-algebra endowed with a family of generators, the so-called **cluster variables**, grouped into overlapping subsets of fixed finite cardinality, which are constructed recursively using mutations. If the set of cluster variables is finite, we speak of **cluster-finite cluster algebras**.*

Theorem 2. *The cluster-finite cluster algebras are parametrized by the finite root systems, i.e. by the Dynkin-diagrams above.*

The algorithm

The **knitting algorithm** shows how to get the cluster-finite cluster algebra $\mathscr{A}_\Delta$ corresponding to a Dynkin-diagram Δ. Let's start with the simplest non trivial Dynkin-diagram. $A_2 : 1 \longrightarrow 2$. Now, we construct the so-called **repetition** $\mathbb{Z}A_2$. First, we draw the product $\mathbb{Z} \times A_2$. Then we draw for each arrow $(n,i) \longrightarrow (n,j)$ a new one $(n,j) \longrightarrow (n+1,i)$ and we get the repetition.

$$
\begin{tikzcd}
\cdots & & (0,2) \arrow[dr] & & (1,2) \arrow[dr] & & (2,2) \arrow[dr] & & (3,2) \arrow[dr] & & (4,2) & \cdots \\
& (0,1) \arrow[ur] & & (1,1) \arrow[ur] & & (2,1) \arrow[ur] & & (3,1) \arrow[ur] & & (4,1) \arrow[ur] &
\end{tikzcd}
$$

Now, we assign to each vertex of the repetition a cluster variable. We start with x_1 and x_2 at the 0-th copy of A_2.

$$
\begin{tikzcd}
\cdots & & x_2 \arrow[dr] & & x_2' \arrow[dr] & & x_2'' \arrow[dr] & & \circ \arrow[dr] & & \circ & \cdots \\
& x_1 \arrow[ur] & & x_1' \arrow[ur] & & x_1'' \arrow[ur] & & x_1''' \arrow[ur] & & \circ \arrow[ur] &
\end{tikzcd}
$$

We can construct $x_1', x_2', x_1'', \ldots$ by "knitting" from the left to the right. To compute x_1', we divide the sum of the immediate predecessor of x_1' and 1 by the left translate of x_1'. That gives us: $x_1' = \frac{x_2+1}{x_1}$. We do the same procedure for $x_2', x_1'', \ldots$. So, finally we get: $x_2' = \frac{x_1+x_2+1}{x_1x_2}$, $x_1'' = \frac{x_1+1}{x_2}$, $x_2'' = x_1$ and $x_1''' = x_2$. Obviously, from now on the whole thing will repeat. Thus, we have 5 cluster variables x_1, x_2, x_1', x_2' and x_1''. So, the cluster algebra $\mathscr{A}_{A_2}$ is a $\mathbb{Q}$-subalgebra of $\mathbb{Q}(x_1, x_2)$ generated by the cluster variables above. We can see that all denominators of the cluster variables are monomials. This can be generalized:

(1) All denominators of all cluster variables are monomials. This holds for all cluster algebras. It's called **Laurent phenomen**.

(2) The computation is **periodic** and we always get only finitely many cluster variables (as expected). Furthermore, the algorithm can easily be extended from Dynkin-diagrams to arbitrary trees, and then one can characterize the Dynkin diagrams by the periodicity of the algorithm.

(3) We have $5 = 2+3$ cluster variables. 2 initial ones x_1, x_2 and 3 non initial ones x_1', x_2' and x_1''. The non initial ones are in natural bijection with the positive roots of the corresponding root system of A_2. To see that there is a bijection, consider the denominators of the cluster variables.

$$\frac{\cdot}{x_1^{d_1} x_2^{d_2}} \mapsto d_1\alpha_1 + d_2\alpha_2$$

Fomin-Zelevinsky showed that this generalizes to arbitrary Dynkin-diagrams. In general, the number of cluster variables in the cluster algebra $\mathscr{A}_\Delta$ is equal to the sum of the rank of Δ and the number of positive roots of Δ.

More examples (for $\Delta = A_3$ and $\Delta = G_2$) are in the lecture notes of Bernhard Keller.