Cluster Algebras Seminar:
Quivers and Cluster Algebras I

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In this seminar we will present and construct the *cluster algebra* associated with a quiver (which is equivalent to the cluster algebra associated with an antisymmetric matrix, see Definition 2; however, working with quivers allows a visual representation of our operations and is therefore more suitable). For this purpose, we introduce the concept of *Seed* and *Mutation* [1], giving some examples [4]. We further present some aspects of the *Path-Algebra* associated with a quiver [2, 3].

Basic Definitions

Definition 1 (Quiver) *A* Quiver *is an oriented graph, described by a quadruple $Q = (Q_0, Q_1, s, t)$: Q_0 is the set of nodes (vertices), Q_1 is the set of directed arcs (arrows) and s, t the functions that map each arrow to its starting (resp. ending) vertex. A quiver Q is called* finite *if both Q_0 and Q_1 have finite cardinality. The* underlying graph of *Q is obtained from Q by forgetting the orientation of the arrows: if this undirected graph is connected (every node is "reachable"), we call the original quiver Q* connected.

As in every directed graph, we define a path of length *l in Q to be a sequence of l arrows $e_1, \dots, e_l$ with $t(e_i) = s(e_{i+1})$ $(1 \leq i < l)$. If additionally we have $t(e_l) = s(e_1)$, we call such a closed path an l-*cycle. *For $l = 1$, i.e. an arrow e with $s(e) = t(e)$, we call this arrow a* loop. ∎

We usually label the nodes with $\{1, 2, \dots, n\}$. By drawing quivers, an integer number k on a single arrow, $i \xrightarrow{k} j$, represents the effective number of arrows that go from i to j (this convention allows us to keep the picture readable and turns out to be very useful later, see Definition 4). If $k < 0$, then this means that $-k > 0$ arrows go from j to i, i.e. $i \xrightarrow{-4} j$ is the same as $i \xleftarrow{4} j$.

Definition 2 (Matrix of a Quiver) *To each quiver Q without loops or 2-cycles there corresponds an $n \times n$ antisymmetric matrix B (i.e. $B^T = -B$) with integer coefficients, where the element $b_{i,j}$ is given as:*

$$b_{i,j} := \#\{\text{arrows } i \to j\} \quad - \quad \#\{\text{arrows } j \to i\}\,.$$

There is a bijection between the set of such quivers and the set of integer antisymmetric matrices: the absence of loops is necessary to ensure the diagonal elements of the matrix to be 0, allowing antisymmetry, while the absence of 2-loops ensures that the matrix is well-defined. ∎

Seeds and their Mutations

Definition 3 (Seed) *A* seed *is a pair (R, u), where:*

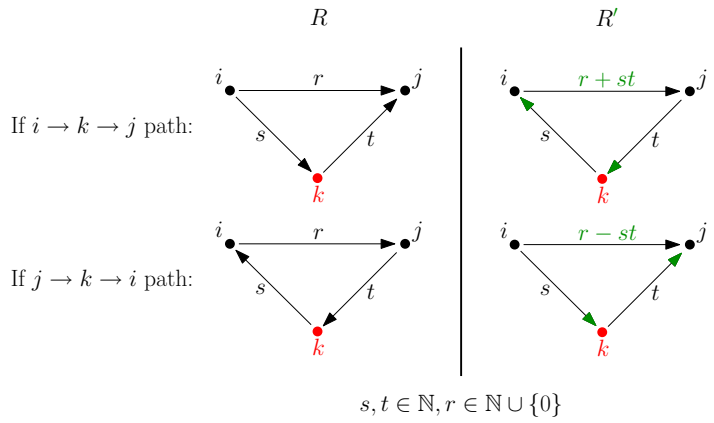
- *R is a finite quiver without loops or 2-cycles, with vertex-set $\{1, \dots, n\}$.*
- *$u = \{u_1, \dots, u_n\}$ is a* free generating set *of the field $\mathbb{Q}(x_1, \dots, x_n)$ of all fractions of $\mathbb{Q}-$polynomials in n variables.*

Given a finite quiver Q without loops or 2-cycles, the initial seed *of Q consists of the seed $(Q, \{x_1, \dots, x_n\})$.* ∎

Notice that in the quiver R of a seed, all the arrows between two vertices point in the same direction (no 2-cycles).

Definition 4 (Mutation of a Seed) *We define a new operation $\mu_k(\cdot)$, called* mutation*, which transforms a given seed into another one by following some easy rules. Let a seed (R, u) be given and fix a vertex k of R. The mutation $\mu_k(R, u) =: (R', u')$ is defined as follows:*

- *R' is obtained from R by modifying the arrows in the following way:*
  - *Reverse all arrows incident with k.*
  - *For every pair of vertices i, j distinct from k, such that both i and j have at least an arrow incident with k, building a directed path $i \to k \to j$ or $j \to k \to i$ (i.e. not both sets of arrows pointing towards k or both starting in k), modify the number of arrows between i and j (even if 0) as shown in the following picture (recall the convention: a negative integer $k < 0$ over an arrow represents a number $-k > 0$ of arrows in the opposite direction, while $k = 0$ means that no arrow is present!).*



- *u' is obtained from u by just replacing the single element u_k with:*

$$u'_k := \frac{1}{u_k}\left(\prod_{\text{arrows } i\to k} u_i + \prod_{\text{arrows } k\to j} u_j\right),$$

where we define the product over an empty set to be equal to 1. ∎

Definition 5 (Mutation Class, Mutation Equivalence) *Given a seed (R, u), the set of all possible quivers obtained from the initial quiver R by any sequence of mutations is called the* mutation class *of R. Two quivers Q_1, Q_2 are called* mutation equivalent *if they belong to the same mutation class (i.e. starting from one of them, it is possible to obtain the other one by a finite sequence of mutations).* ∎

We notice that (R', u') is again a seed, and that a mutation is an involution, i.e. $\mu_k(\mu_k(R, u)) = (R, u)$.

It is interesting to notice that a mutation at a sink/source vertex (i.e. without incoming/outgoing arrows) simply reverses the sense of the arrows of this mutating vertex. From this follows that all the orientations of a tree are mutation equivalent.

As an additional remark, which will be useful later, we notice that a mutation performed on an orientation of a Dynkin diagram can lead to a quiver which is not a Dynkin diagram anymore.

According to the notation of Definition 2, the matrix associated to the resulting mutated quiver R' is given by:

$$b'_{i,j} = \begin{cases} -b_{i,j}, & \text{if } i = k \text{ or } j = k \\ b_{i,j} + \operatorname{sgn}(b_{i,k}) \cdot \max\{b_{i,k}b_{k,j}, 0\}, & \text{otherwise} \end{cases}$$

For some examples of seed mutations see the blackboard, or [1, 4].

The Cluster Algebra

Definition 6 (Cluster Algebra) *Let Q be a finite quiver without loops or 2-cycles with vertex-set $\{1, \ldots, n\}$ and consider the initial seed $(Q, \{x_1, \ldots, x_n\})$. We define:*

- *The **Clusters** with respect to Q as all the sets u appearing in seeds (R, u), obtained from the initial seed by mutations (these sets all have the same cardinality n and can be overlapping);*
- *The **Cluster Variables** for Q to be the elements of all clusters;*
- *The **Cluster Algebra $\mathcal{A}_Q$** to be the $\mathbb{Q}-$subalgebra of the field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all the cluster variables.*

∎

Theorem (Gekhtman-Shapiro-Vainshtein) Each cluster u occurs in a unique seed (R, u). ∎

Thus, the cluster algebra consists of all $\mathbb{Q}-$linear combinations of monomials in the cluster variables. Since $\mathcal{A}_Q$ is generated by its cluster variables, it suffices to produce the list of these variables in order to describe $\mathcal{A}_Q$.

In the previous seminar of last week we have seen that for some special quivers, namely the oriented versions of a Dynkin diagram Δ, it is possible to directly construct the cluster variables (hence describing $\mathcal{A}_\Delta$) without first having to construct the clusters: this was done by a much easier procedure, the *Knitting Algorithm*, without introducing the concept of mutation. With Definitions 4 and 6 we have now generalized this concept, producing an important additional structure not provided by the knitting algorithm: the clusters.

Next week we will see that the number of cluster variables of $\mathcal{A}_Q$ is finite if and only if the quiver Q is mutation equivalent to (an orientation of) a Dynkin diagram.

Definition 7 (Exchange Graph) *The* exchange graph *associated with Q, which graphically represents the recursive construction of $\mathcal{A}_Q$ through mutations, is the undirected graph whose vertices are the different obtainable seeds (up to renumbering of the vertices and the associated cluster variables) and the edges represent the mutations (the edges are undirected, since a mutation is an involution!).*

The Path Algebra

Notation: Recall the notion of path given in Definition 1. A path of length l with arrows $\alpha_1, \ldots, \alpha_l$, *source* $s(\alpha_1) = a$ and *target* $t(\alpha_l) = b$ will be denoted by $(a \mid \alpha_1, \ldots, \alpha_l \mid b)$. The *trivial (or stationary) path* of length 0 is denoted by $\varepsilon_a := (a \mid\mid a)$.

Definition 8 (Path Algebra) *Let Q be any quiver (also with loops and 2-cycles). The* path algebra *KQ of Q is the $K-$algebra whose underlying $K-$vector space has as its basis the set of all paths $(a \mid \alpha_1, \ldots, \alpha_l \mid b)$ of length $l \geq 0$ in Q and such that the product of two basis vectors $(a \mid \alpha_1, \ldots, \alpha_l \mid b)$ and $(c \mid \beta_1, \ldots, \beta_L \mid d)$ is defined as follows:*

$$(a \mid \alpha_1, \ldots, \alpha_l \mid b)(c \mid \beta_1, \ldots, \beta_L \mid d) := \delta_{b,c} \cdot (a \mid \alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_L \mid d), \text{ where } \delta_{b,c} = \begin{cases} 1, & \text{if } b = c \\ 0, & \text{otherwise} \end{cases}. \quad ∎$$

We remark that KQ can be decomposed in a direct sum $KQ = (KQ)_0 \oplus (KQ)_1 \oplus \ldots \oplus (KQ)_n \oplus \ldots$, where $(KQ)_n$ is the subspace of KQ generated by the set of all paths in Q of length n. It holds: $(KQ)_n \cdot (KQ)_m \subseteq (KQ)_{n+m}$.

Theorem *(page 45 in [2])*
Let Q be a quiver and KQ its path algebra. Then:

(a) KQ is an associative algebra.

(b) KQ has an identity element $\Leftrightarrow$ Q_0 is finite.

(c) KQ is finite dimensional $\Leftrightarrow$ Q is finite and acyclic. ∎

Theorem *(pages 46,47 in [2])*
Let Q be a *finite* quiver and KQ its path algebra. Then:

(a) KQ is connected $\Leftrightarrow$ Q is connected.
(an algebra is called connected, if it can not be expressed as the product of two different algebras)

(b) The element $1 = \sum_{a \in Q_0} \varepsilon_a$ is the identity and the set $\{\varepsilon_a \mid a \in Q_0\}$ of all the stationary paths $\varepsilon_a = (a \mid\mid a)$ is a complete set of primitive orthogonal idempotents for KQ. ∎

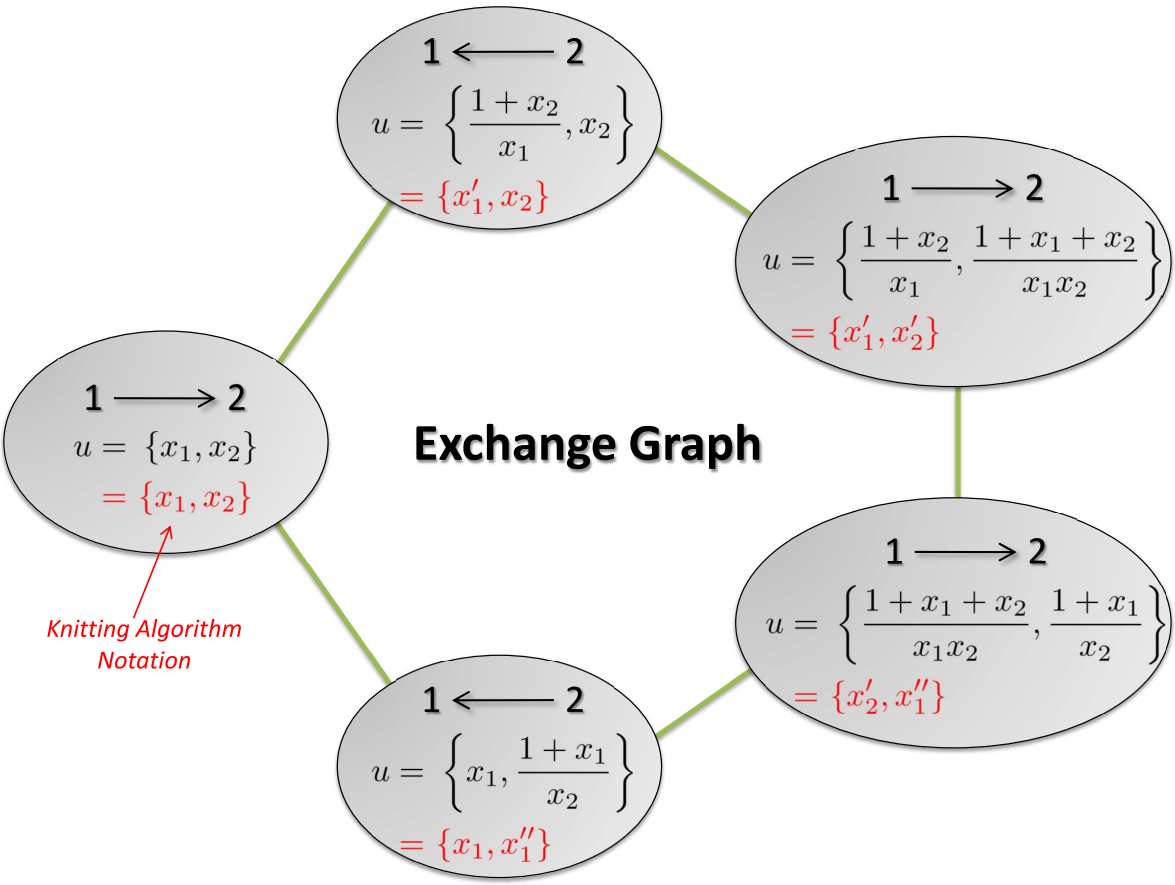
References

[1] Bernhard Keller, *Cluster algebras, quiver representations and triangulated categories*, May 2009 (*http://front.math.ucdavis.edu/0807.1960*).

[2] Ibrahim Assem, Andrzej Skowronski, Daniel Simson, *Elements of the Representation Theory of Associative Algebras*, Cambridge University Press, 2006.

[3] Lecture Notes of the seminar *"Darstellungstheorie von Köchern"*, held by Karin Baur and Ivo dell'Ambrogio, 2008 (*http://www.math.ethz.ch/education/bachelor/seminars/hs2008/dtk*).

[4] Bernhard Keller, *A Java Applet for the Mutation Rule*, *http://people.math.jussieu.fr/ keller/quivermutation*.



Exchange Graph (Compact Notation)