Cluster algebras II and remarkable phenomena

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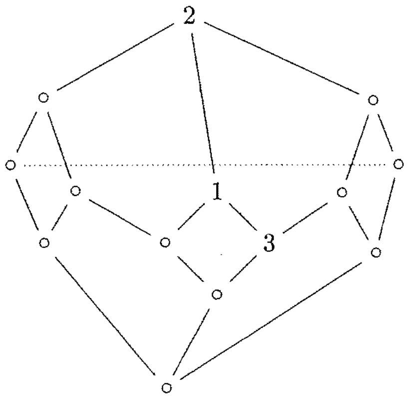
1 Cluster algebras with finitely many cluster variables

Let us consider the quiver

$$Q : 1 \longrightarrow 2 \longrightarrow 3$$

obtained by endowing the Dynking diagram A_3 with a linear orientation. By applying the recursive construction to the initial seed $(Q, \{x_1, x_2, x_3\})$ one finds exactly fourteen seeds (modulo simultaneous renumbering of vertices and cluster variables). These are the vertices of the exchange graph, which is isomorphic to the third Stasheff associahedron.



The vertex labeled 1 corresponds to $(Q, \{x_1, x_2, x_3\})$, the vertex labeled 2 to $\mu_2(Q, \{x_1, x_2, x_3\})$, which is given by

$$\begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd} \;,\; \{x_1, \frac{x_1 + x_3}{x_2}, x_3\}\,,$$

and the vertex 3 to $\mu_1(Q, \{x_1, x_2, x_3\})$, which is given by

$$\begin{tikzcd} 1 & 2 \arrow[l] \arrow[r] & 3 \end{tikzcd} \;,\; \{\frac{1 + x_2}{x_1}, x_2, x_3\}.$$

As expected (see first talk), we find a total of $3 + 6 = 9$ cluster variables, which correspond bijectively to the faces of the exchange graph. The clusters $\{x_1, x_2, x_3\}$ and $\{x_1', x_2, x_3\}$, where $x_1' = \frac{1+x_2}{x_1}$, also appear naturally as slices of the repetition, where by a *slice*, we mean a full connected subquiver containing a representative of each orbit under the horizontal translation (a subquiver is *full* if, with any two vertices, it contains all the arrows between them).

$$\begin{tikzcd}
\circ \arrow[dr] & & x_3 \arrow[dr] & & x_3' \arrow[dr] & & x_1 \arrow[dr] & & x_3' \arrow[dr] & \\
& x_2 \arrow[ur] \arrow[dr] & & x_2' \arrow[ur] \arrow[dr] & & x_2'' \arrow[ur] \arrow[dr] & & x_2 \arrow[ur] \arrow[dr] & & x_2' \\
x_1 \arrow[ur] & & x_1' \arrow[ur] & & x_1'' \arrow[ur] & & x_1''' \arrow[ur] & & x_3 \arrow[ur] &
\end{tikzcd}$$

In fact, as it is easy to check, each slice yields a cluster. However, some clusters do not come from slices, for example the cluster $\{x_1, x_3, x_1''\}$, where $x_1'' = \frac{x_1+x_3}{x_2}$, associated with the seed $\mu_2(Q, \{x_1, x_2, x_3\})$. The phenomena observed in this examples are explained by the following key theorem. Note that a Dynkin diagram is simply laced if it contains only simple links, i.e. it is of Type A_n, D_n, E_6, E_7 or E_8.

Theorem 1.1 (Fomin-Zelevinsky). *Let Q be a finite connected quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$. Let $\mathscr{A}_Q$ be the associated cluster algebra.*

a) All cluster variables are Laurent polynomials, i.e. their denominators are monomials.

b) The number of cluster variables is finite iff Q is mutation equivalent to an orientation of a simply laced Dynkin diagram Δ. In this case, Δ is unique and the non initial cluster variables are in bijection with the positive roots of Δ; namely, if we denote the simple roots by $\alpha_1, \ldots, \alpha_n$, then for each positive root $\sum d_i \alpha_i$, there is a unique non initial cluster variable whose denominator is $\prod x_i^{d_i}$.

c) The knitting algorithm yields all cluster variables iff the quiver Q has two vertices or is an orientation of a simply laced Dynkin diagram Δ.

It is not hard to check that the knitting algorithm yields exactly the cluster variables obtained by iterated mutations at sinks and sources. Remarkably, in the Dynkin case, all cluster variables can be obtained in this way.

Remark 1.1. *The construction of the cluster algebra shows that if the quiver Q is mutation-equivalent to Q', then we have an isomorphism*

$$\mathscr{A}_{Q'} \xrightarrow{\sim} \mathscr{A}_Q$$

preserving clusters and cluster variables. Thus, to prove that the condition in b) is sufficient, it suffices to show that $\mathscr{A}_Q$ is cluster-finite if the underlying graph of Q is a Dynkin diagram.

In general it is unknown how to decide whether two given quivers are mutation-equivalent. However, for certain restricted classes, the answer to this problem is known: Trivially, two quivers with two vertices are mutation-equivalent iff they are isomorphic. But it is already a non-trivial problem to decide when a quiver

$$\begin{tikzcd} 1 \arrow[rr, "r"] & & 2 \arrow[dl, "t"] \\ & 3 \arrow[ul, "s"] & \end{tikzcd},$$

where r, s and t are non negative integers, is mutation-equivalent to a quiver without a 3-cycle: One can show that this is the case iff the Markoff inequality

$$r^2 + s^2 + t^2 - rst > 4$$

holds or one among r, s and t is < 2.

Example 1.1. *One can check that for $3 \leq n \leq 8$, the following quiver glued together from $n-2$ triangles*

$$\begin{tikzcd} 1 \arrow[d] & 3 \arrow[l] \arrow[d] & 5 \arrow[l] \arrow[d] & \cdots \arrow[l] & n-1 \arrow[l] \arrow[d] \\ 2 \arrow[ur] & 4 \arrow[l] \arrow[ur] & 6 \arrow[l] \arrow[ur] & \cdots \arrow[l] \arrow[ur] & n \arrow[l] \end{tikzcd}$$

is cluster-finite of respective cluster-type A_3, D_4, D_5, E_6, E_7 and E_8 and that it is not cluster-finite if $n > 8$.

2 Associahedron

First some definitions that we all know from geometry:

Definition 2.1. *A* ***polygon*** *P is a closed plane figure, that is bounded by a closed path composed of a finite sequence of straight line segments (sides). It is called* ***simple,*** *if its sides do not cross and it is called* ***convex*** *if it is simple and its interior is a convex set.*

Definition 2.2. *A* ***triangulation*** *of a polygon P is the decomposition of P into a set of* ***non-overlapping*** *triangles whose union is P.*

Given a convex $(n+3)$-gon P, i.e. a polygon with $n+3$ sides.

Lemma 2.1. *Each triangulation of P involves exactly n diagonals.*

Theorem 2.1. *The number of possible triangulations of a convex $(n+3)$-gon is given by the* ***Catalan number***

$$\frac{1}{n+2}\binom{2n+2}{n+1}.$$

Example 2.1. *1. $n = 1$: 2 triangulations of a convex quadrilateral (four-sided figure),*

2. $n = 2$: 5 triangulations of a pentagon,

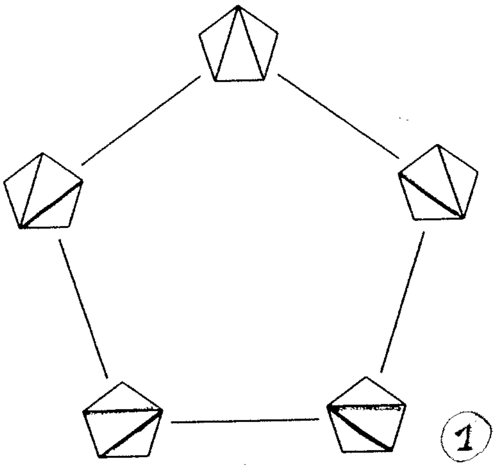
3. $n = 3$: 14 triangulations of a hexagon.

Given a triangulation of P we can get another triangulation by removing a diagonal to create a quadrilateral, then replace the removed diagonal with the other diagonal of the quadrilateral. This procedure is called **diagonal flip**.

Definition 2.3. *We call the graph defined by diagonal flips the* ***exchange graph*** *of P.*

Example 2.2. *1. $n = 2$: The exchange graph for triangulations of a pentagon,*

2. $n = 3$: The exchange graph for triangulations of a hexagon.



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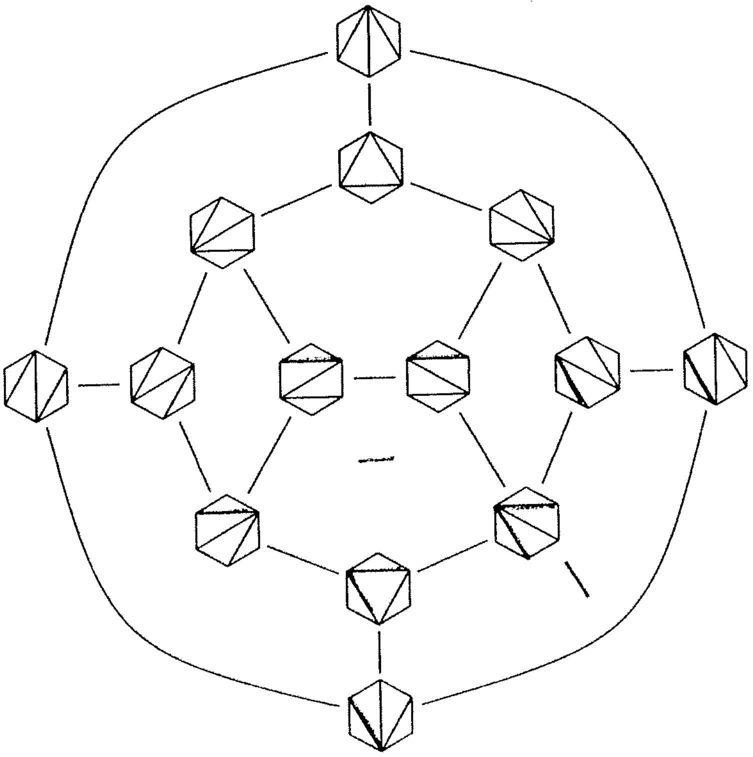

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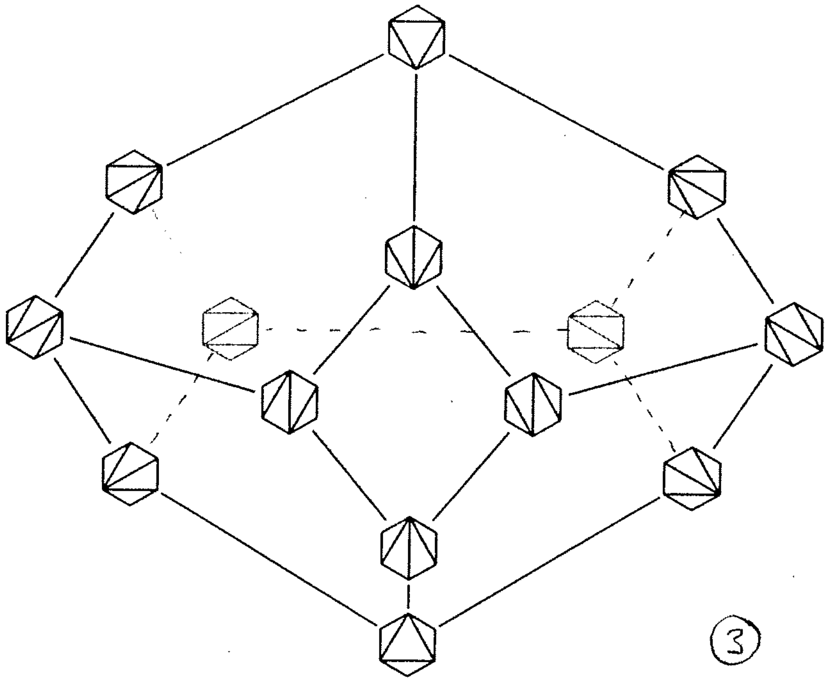
The drawing of the exchange graph in figure 2 fails to convey its crucial property: this exchange graph is the 1-skeleton of a convex polytope, the 3-dimensional associahedron.



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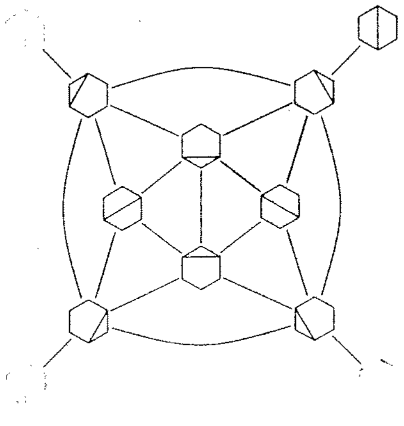
In order to formally define the n-dimensional associahedron associated to a $(n+3)$-gon, we start by describing the object which is dual to it.

Definition 2.4 (The **dual complex** of an associahedron). *Consider the following simplicial complex:*

- *vertices: diagonals of a convex* $(n+3)$*-gon,*
- *simplices: partial triangulations of the* $(n+3)$*-gon (collections of non-crossing diagonals)*
- *maximal simplices: triangulations of the* $(n+3)$*-gon (collections of* n *non-crossing diagonals)*

In other words we connect two edges if the diagonals do not cross. Below we see the simplicial complex dual to the 3-dimensional associahedron.



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Theorem 2.2. *The simplicial complex described in Definition 2.4 can be realized as the boundary of an n-dimensional convex polytope.*

Definition 2.5 (The associahedron). *The n-dimensional* ***associahedron*** *is the convex polytope that is dual to the polytope of Theorem 2.2.*

3 Matrix mutations

Given a convex $(n+3)$-gon P. Fix a triangulation T of P. Label the diagonals of T arbitrarily by the numbers $1, ..., n$ and label the $n+3$ sides of P by the numbers $n+1, ..., 2n+3$. The combinatorics of T can be encoded by the **edge-adjacency matrix** $\tilde{B} = (b_{ij})$. This is the $(2n+3) \times n$ matrix whose entries are given by

$$b_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ label two sides in some triangle of } T \text{ so that } j \\ & \text{follows } i \text{ in the clockwise traversal of the triangle's boundary,} \\ -1 & \text{if the same holds, with the counter-clockwise order,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the first index i is a label for a side or a diagonal of the $(n+3)$-gon, while the second index j must label a diagonal. The **principal part** of $\tilde{B}$ is the $n \times n$ submatrix $B = (b_{ij})_{i,j \in \{1,..,n\}}$.
In the language of matrices $\tilde{B}$ and B, diagonal flips can be described as certain transformations called matrix mutations.

Definition 3.1. *Let $B = b_{ij}$ and $B' = (b'_{ij})$ be integer matrices. We say that B' is obtained from B by* ***matrix mutation*** *in direction $k \in \mathbb{N}$, and write $B' = \mu_k(B)$, if*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i,j\}, \\ b_{ij} + |b_{ik}|\, b_{kj} & \text{if } k \notin \{i,j\} \text{ and } b_{ik}b_{kj} > 0, \\ b_{ij} & \text{otherwise.} \end{cases}$$

It is easy to check that a matrix mutation is an involution, i.e. $\mu_k(\mu_k(B)) = B$.

Theorem 3.1. *Let P be a $(n+3)$-gon. Assume that $\tilde{B}$ and $\tilde{B}'$ (resp. B and B') are the edge-adjacency matrices (resp. their principal parts) for two triangulations T and T' of P obtained from each other by flipping the diagonal numbered $k \in \{1, ..., n\}$; the rest of the labels are the same in T and T'. Then*

$$\tilde{B}' = \mu_k(\tilde{B}) \text{ (resp. } B' = \mu_k(B)).$$

4 Exchange relations

In the previous section we saw how we can transform a triangulation into another by matrix multiplications. Now we shall introduce algebraic transformations that are compatible with the matrix multiplication. Let us consider a fixed initial triangulation T_o of a convex $(n+3)$-gon . Further introduce a variable for each diagonal of this triangulation, also for each side of the $(n+3)$-gon. So all in all we need $2n+3$ variables. Now we associate a rational function in this $2n+3$ variables to every diagonal of the $(n+3)$-gon. Doing a diagonal flip all but one rational function associated to the current triangulation remain unchanged:

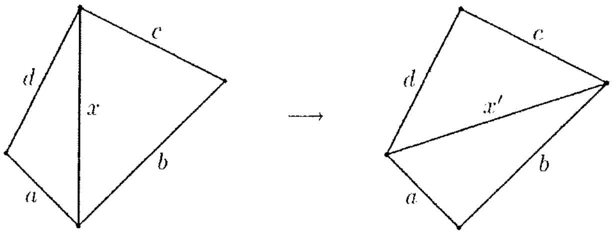


Figure 3.14. A diagonal flip

This function x associated with the diagonal being removed gets replaced by a rational function $x^{'}$ associated with the resulting diagonal, where $x^{'}$ is determined from the **exchange relation**

$$xx^{'} = ac + bd$$

Lemma 4.1. *The rational function x_γ associated to each diagonal γ does not depend on the particular choice of a sequence of flips that connects the initial triangulation with another one containing γ.*

Let us now illustrate this lemma with the triangulation of a pentagon, that is we set $n=2$. As shown on the next figure we labeled the sides of the pentagon by the variables $q_1, q_2, ..., q_5$.

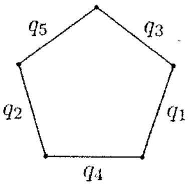
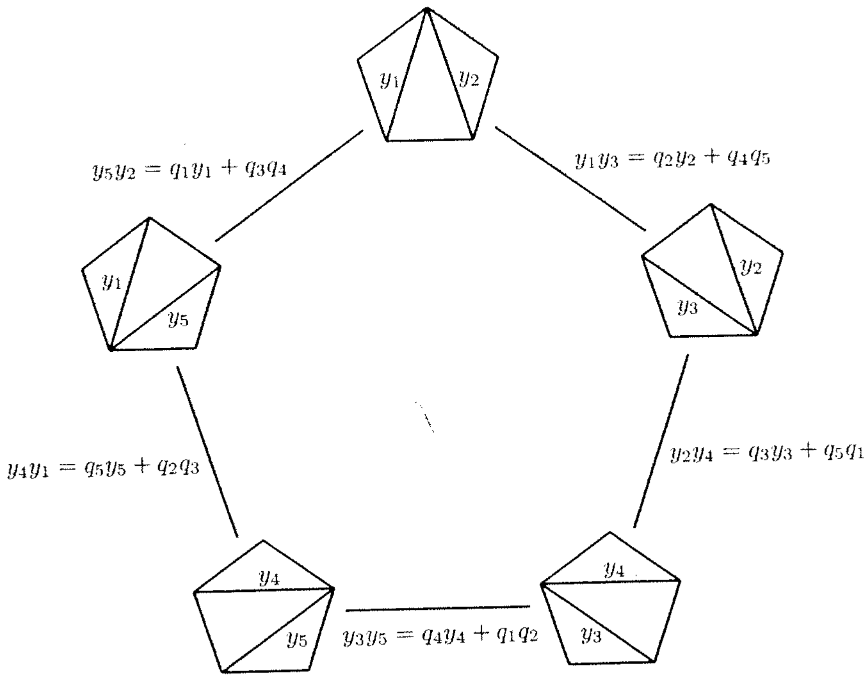


Figure 3.15. Labeling the sides of a pentagon

Consider the following illustration of the transformations of the triangulations of the pentagon.



Starting at the top of the last figure. Let us now express $y_3, y_4, \ldots$ in terms of y_1, y_2 and $q_1, \ldots, q_5$:

$$y_3 = \frac{q_2y_2 + q_4q_5}{y_1}$$

$$y_4 = \frac{q_3y_4 + q_5q_1}{y_2} = \frac{q_3q_2y_2 + q_3q_4q_5 + q_5q_1y_1}{y_1y_2}$$

$$y_5 = \frac{q_4y_4 + q_1q_2}{y_3} = \frac{q_3q_4 + q_1y_1}{y_2}$$

Finally we see that

$$y_1 = \frac{q_5y_5 + q_2q_3}{y_4} = \ldots = y_1$$

$$y_2 = \frac{q_1y_1 + q_3q_4}{y_5} = \ldots = y_2$$

The last two equations verifies Lemma 4.1, which we can now rephrase by saying that there are no "monodromies" associated with sequences of flips that begin and end at the same triangulation.

5 Seed and clusters

Consider a diagonal flip transforming a triangulation T of a convex $(n+3)$-gon into a triangulation $T^{'}$, as we visualized in the first figure of the section Exchange relations. The corresponding exchange relation, as described above, can be written entirely in terms of the edge-adjacency matrix $\tilde{B}$. As before we label the diagonals of the triangulation T in some way by numbers $1, 2, ..., n$ and for the sides of the $(n+3)$-gon we continue by labeling them from $n+1$ to $m = 2n+3$. We get the new triangulation $T^{'}$ is the same except for the one diagonal (say, labeled k) that is getting getting exchanged between T and $T^{'}$. This labeling of T allows us to denote the associated rational functions by $x_1, ..., x_m$. For $T^{'}$, we get the same rational functions except that x_k is replaced by $x_k^{'}$. The the exchange relation under consideration takes the form

$$x_k x_k^{'} = \prod_{b_{ik}>0, 1\leq i\leq m} x_i^{b_{ik}} + \prod_{b_{ik}<0, 1\leq i\leq m} x_i^{-b_{ik}}$$

The right-hand side of this equation is the sum of two monomials whose exponents are the absolute values of the entries in the k-th column of $\tilde{B}$, while the sign of an entry determines which monomial the corresponding term contribute to. All in all we can encode the combinatorics of flips and the algebra of exchange relations entirely in terms of the matrices $\tilde{B}$ using, first the machinery of matrix mutations and, second the "birational dynamics " given by the last equation.

Definition 5.1. *A cluster algebra $\mathcal{A}$ is a commutative ring contained in an ambient field $\mathcal{F}$ isomorphic to the field of rational functions in m variables over $\mathbb{Q}$.*

$\mathcal{A}$ is generated inside F by a set of generators. These generators are obtained from an initial **seed** via an iterative process of seed mutations which follows a set of canonical rules.

Definition 5.2. *A seed in $\mathcal{F}$ is a pair $(\tilde{x}, \tilde{B})$, where*

- *$\tilde{x} = \{x_1, ..., x_m\}$ is a set of m algebraically independent generators of $\mathcal{F}$, which is split into a disjoint union of an n-element cluster $x = \{x_1, ..., x_n\}$ and an $(m-n)$-element set of frozen variables $c = \{x_{n+1}, ..., x_m\}$;*
- *$\tilde{B} = (b_{ij})$ is an $m \times n$ integer matrix of rank n whose principal part $B = (b_{ij})_{i,j\in[n]}$ is skew-symmetrizable, i.e., there exists a diagonal matrix D with positive diagonal entries such that DBD^{-1} is skew-symmetric.*