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THE CLASSIFICATION PROBLEM
REPRESENTATION-FINITE, TAME AND WILD QUIVERS

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In the first part of this week's session of the seminar on *Cluster Algebras* by Prof. Baur we sketch some results of quiver representation, following the first cap of chapter five of this seminar's principal note by B. Keller [6]. We denote by Q a finite quiver over an algebraically closed field k with underlying graph $\overline{Q}$.

Let us recall that any non-zero representation of a finite quiver Q without oriented cycles can be decomposed into a finite direct sum of indecomposable representations (representations that are not isomorphic to the direct sum of two non-zero representations). By the Krull-Remark-Schmidt theorem, the indecomposables appearing in such a decomposition are unique up to order and isomorphisms. Thus, if we understand all the indecomposables of a representation of a quiver Q and on top the morphisms acting between them (seen as a matrix), that is if we can give a complete list of pairwise non-isomorphic indecomposable representations, we fully understand the category of representations of Q. This is the classification problem.

Definition. *The quiver Q is called* representation-finite *if, up to isomorphisms, it has only finitely many indecomposable representations. If not, it is called* representation-infinite.

By the results presented last week, we fully understand the category of representations $\mathrm{rep}_k Q$ in case Q is representation-finite by Gabriel's theorem. The question arising is whether there exists some classification for quivers of infinite representation type. We will see that this is indeed the case for some representation-infinite quivers but that the problem is rather delicate in general! We split the group of representation-infinite quivers into tame and wild ones.

Definition. *Let Q be representation-infinite.*

i) Q is called tame *if the problem of classifying its representations is tame, that is if the indecomposable representations in every dimension occur in a finite number of one-parameter families.*

ii) Q is called wild *if the problem of classifying its representations is wild, that is if the indecomposable representations occur in families of at least two parameters.*

One may wonder if the tame and wild terminologies are to be understood as aptronyms. This turns out to be the case as there exists a theory for the classification problem of tame quivers by well-understood objects while the wild classification problem turns out to contain the problem of classifying any pairs of matrices up to simultaneous similarity; thus wild problems are indeed hopeless in a certain sense.

We briefly recall the results seen for representation-finite quivers and then go along with quivers of infinite representation type.

1 Quivers of finite representation type and Gabriel's theorem

Theorem 1 (Gabriel)**.** *Let Q be a finite connected quiver without oriented cycles. The following are equivalent:*

i) Q is representation-finite.

ii) $\overline{Q}$ is a simply laced Dynkin diagram.

iii) The quadratic form q_Q is positive definite where

$$q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}, \quad x \in \mathbb{Z}^n.$$

Moreover, in the case where Q is representation-finite (especially without loops or 2-cycles), we cannot only describe it's category of representation but also link it to the cluster algebra. A first step of such a connection is illustrated by the following bijective correspondance.

$$\left\{ \begin{array}{c} \text{positive roots} \\ \text{of the} \\ \text{quadratic form } q_Q \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of indecomposable} \\ \text{representations } V \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{non-initial} \\ \text{cluster variables } X_V \end{array} \right\}$$

A proof of the first bijection is given by Gabriel. The second statement follows by Gabriel's theorem in combination with a result presented at the very beginning of this seminar. Namely that for each positive root $\alpha = \sum_{i=1}^n d_i \alpha_i$ of the finite root system that corresponds to the Dynkin diagram $\overline{Q}$ there exists a unique non initial cluster variable

$$X_\alpha = \frac{*}{x_1^{d_1} \cdots x_n^{d_n}}$$

where (d_i) is the dimension vector of the indecomposable representation.

Example 1. *In the case where Q is the Dynkin quiver A_2 with vertices x_1, x_2,*

$$x_1 \xrightarrow{1} x_2$$

we found the non-initial cluster variables x_1', x_2', x_1'' applying the Knitting algorithm to x_1 and x_2. Considering the indecomposable representation

$$k \xrightarrow{1} k$$

of A_2 with indecomposable subrepresentations S_1, S_2, P_1, we get the following bijective correspondance (also mentioned last week):

$$S_1 = (k \longrightarrow 0) \xleftrightarrow{\cong} X_{S_1} = \tfrac{1+x_2}{x_1} \qquad (= x_1')$$

$$S_2 = (0 \longrightarrow k) \xleftrightarrow{\cong} X_{S_2} = \tfrac{1+x_1}{x_2} \qquad (= x_2')$$

$$P_1 = (k \longrightarrow k) \xleftrightarrow{\cong} X_{P_1} = \tfrac{x_1+1+x_2}{x_1 x_2} \qquad (= x_1'')$$

2 A Classification Theorem for Tame Quivers

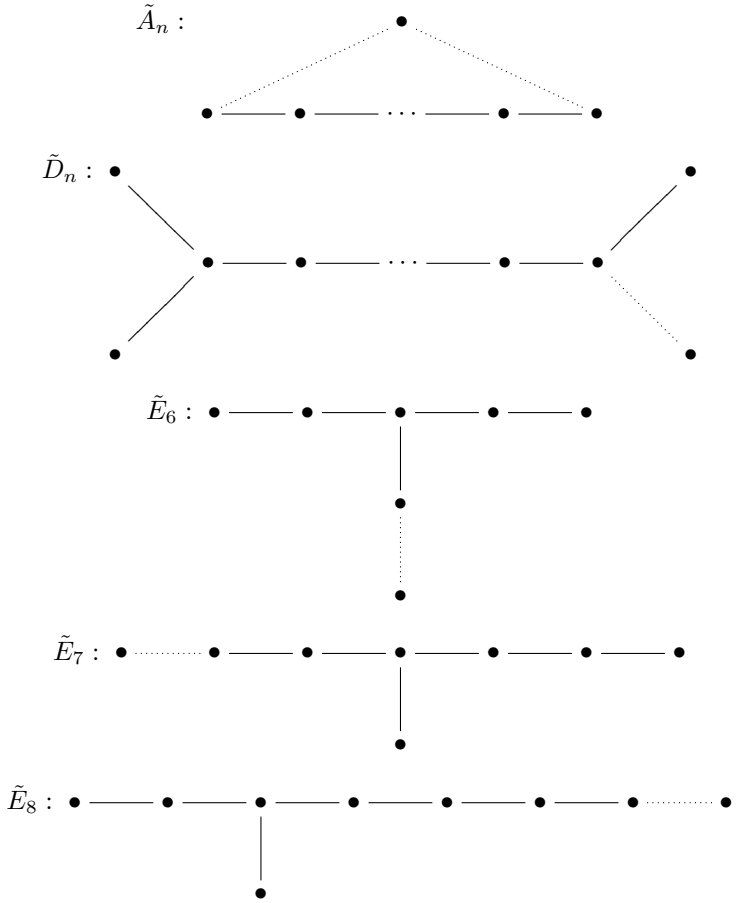
Theorem 2. *Let Q be a finite connected quiver without oriented cycles. The following are equivalent:*

i) Q is representation-infinite and tame.

ii) $\overline{Q}$ is a simply laced extended Dynkin diagram.

iii) The quadratic form q_Q is positive semi-definite but not positive definite.

The extended Dynkin diagrams are as well called Euclidean diagrams. The number of vertices is the lowered index plus 1 in each case. The punctuated arrow with the additional vertex in each diagram shows the extension from the corresponding Dynkin diagram A_n, D_n, E_6, E_7, E_8, respectively.



A discussion of this theorem and of general tame problems can be found in [2]. It uses homological algebra, the theory on postprojective and preinjective finitely generated modules over the path algebra of a quiver, the local properties of the endomorphism algebra and the correspondance of indecomposables to quotients of the polynomial algebras $k[t]$ in one or even more indeterminantes. It may be remarked that there are made distinctions between tame problems of finite growth, linear growth and polynomial growth, thus the class of tame problems is not homogeneous in general.

Example 2. *i) Consider the Kronecker quiver K_2 with $\overline{K}_2 \cong \tilde{A}_1$:*

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

The pairwise non-isomorphic indecomposable representations up to isomorphism are given by the following representations for $m \in \mathbb{N}$ and $\lambda \in k$ (a calculation is to be found in [1])

$$k^m \underset{J_{m,\lambda}}{\overset{\mathbf{1}_m}{\rightrightarrows}} k^m$$

where

$$\mathbf{1}_m = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \text{ is the } m \times m \text{ identity matrix and}$$

$$J_{m,\lambda} = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

is a Jordan block of size $m \times m$. The number of such indecomposables is infinite, but organizedin a one-parameter family in every dimension. Thus, K_2 is tame.

ii) Even one cannot apply the above theorem to the simple loop L_1

$$\alpha \circlearrowright 1$$

which has only one vertex and one arrow, it can be shown to be representation tame analogously as for K_3. A representation of L_1 is a vector space V together with a linear endomorphism $f : V \to V$, and by the choice of a basis, this is isomorphic to the pair (k^m, F) where F is the corresponding matrix describing f w.r.t. this basis. Two of such representations (k^m, F) and (k^n, G) are isomorphic if and only if $m = n$ and F and G are conjugate. Moreover, the matrix F is conjugate to a Jordan matrix

$$J_{m_1,\lambda_1} \oplus J_{m_2,\lambda_2} \oplus \cdots \oplus J_{m_t,\lambda_t}$$

and consequently

$$(k^m, F) \cong (k^{m_1}, J_{m_1,\lambda_1}) \oplus \cdots \oplus (k^{m_t}, J_{m_t,\lambda_t}).$$

For those interested in representaion-tame problems, we'd like to point out that the definition of tame given at the beginning of this handout is not the original definition but rather a consequence thereof.

Definition 1 (Original definition)**.** *A finite-dimensional k-algebra A is representation-tame if, for all $d \geq 0$, the category* $\mathrm{ind}_d A$ *of indecomposable modules of k-dimension d us is almost parametrized by a finite family of functors*

$$N^1, \ldots, N^{m_d} : \mathrm{ind}_1 k[t] \longrightarrow \mathrm{mod} A.$$

Let $\mu_A(d)$ be the minimal number $m_d \geq 0$ of such an almost parametrization. Then one introduces the following distinction:

- A is of finite growth if $\exists m \geq 0 : \mu_A(d) \leq m \ \forall d \geq 1$.
- A is of linear growth if $\exists m \geq 0 : \mu_A(d) \leq m \times d \ \forall d \geq 1$.
- A is of polynomial growth if $\exists m \geq 0 : \mu_A(d) \leq d^m \ \forall d \geq 1$.

3 The Problem on Wildness

To see a bit the spirit of wildness, we remark that in general representation theory the classification problem is called wild if it contains the problem of classifying pairs of matrices up to simultaneous similarity.

Definition. *i) Square matrices A and B of the same dimension are said to be* similar *if there exists an invertible matrix M such that $AM = MB$. Similarity is then the problem of testing matrix similarity.*

ii) A related problem is simultaneous similarity*, where the question is to test, given pairs of matrices $(A_1, B_1), \ldots, (A_m, B_m)$ of the same dimension, whether there is an invertible matrix M such that for all i, $1 \leq i \leq m$ it holds that $A_i M = M B_i$.*

One can show that the problem of classifying pairs of matrices up to simultaneous similarity contains the classification matrix problems given by quivers and that all problems of classifying representations of wild quivers have the same complexity such that a solution of one implies a solution of the others. For more details on that subject we refer to [4].

We shortly want to illustrate the statement on complexity along an

Example 3. *Consider the extended Kronecker quiver K_3*

$$\begin{tikzcd} 1 \arrow[r, bend left=60, "\alpha"] \arrow[r, "\beta"] \arrow[r, bend right=60, "\gamma"'] & 2 \end{tikzcd}$$

The pairwise non-isomorphic and indecomposable representations are given by $V_{\lambda,\mu}$ where

$$V_{\lambda,\mu}(1) = k^2, \quad V_{\lambda,\mu}(2) = k \text{ and}$$
$$V_{\lambda,\mu}(\alpha) = (1\ 0), \quad V_{\lambda,\mu}(\beta) = (0\ 1), \quad V_{\lambda,\mu}(\gamma) = (\lambda\ \mu).$$

Thus, the pairwise non-isomorphic indecomposables define a two parameter family. By what we mentioned above, it is hopeless to try to write down a complete list of all indecomposables. The extended Kronecker quiver K_3 is wild.

Proposition. *If we could solve the classification problem for the quiver K_3 then we could solve it for any quiver Q.*

The proof to this proposition given in [3] proceeds in three steps.

Step 1. *If we could solve the classification problem for K_3 we could solve it for L_2, the loop with one vertex and two arrows starting and ending at this single vertex.*

$$\begin{tikzcd} 0 \arrow[loop left, "\alpha_1"] \arrow[loop right, "\alpha_2"] \end{tikzcd}$$

Given a representation V of L_2, we set $V_0 := V(0)$ and $V_i := V(\alpha_i)$, $i = 1, 2$. We construct a representation $F(V)$ of K_3 setting

$$F(V)(1) = V_1 \oplus V_2, \quad F(V)(2) = V_1 \text{ and}$$

$$F(V)(\alpha) = (\mathbf{1}_{V_0}\ 0), \quad F(V)(\beta) = (0\ \mathbf{1}_{V_0}), \quad F(V)(\gamma) = (V_1, V_2)$$

$$\begin{tikzcd} V_0 \oplus V_0 \arrow[r, bend left=60, "(\mathbf{1}_{V_0}\ 0)"] \arrow[r, "(0\ \mathbf{1}_{V_0})"] \arrow[r, bend right=60, "(V_1, V_2)"'] & V_0 \end{tikzcd}$$

One can show that $\mathrm{End}_{L_2}(V)$ and $\mathrm{End}_{K_3}(F(V))$ are always isomorphic such that V is indecomposable if and only if $F(V)$ is indecomposable and two representations F and W of L_2 are isomorphic if and only if $F(V)$ and $F(W)$ are isomorphic representations of K_3.

This shows that the classification problem of L_2 is included in the classification problem of K_3.

> **Step 2.** *If we could solve the classification problem for K_3 we could solve it for L_t for any $t \geq 2$.*

For a representation V of L_t define a representation $G(V)$ of L_2 by

$$G(V)(0) = V_0^{t+1},$$

$$G(V)(\alpha_1) = \begin{pmatrix} 0 & \mathbf{1}_{V_0} & & & \\ & 0 & \mathbf{1}_{V_0} & & \\ & & \ddots & \ddots & \\ & & & 0 & \mathbf{1}_{V_0} \\ & & & & 0 \end{pmatrix} \text{ and}$$

$$G(V)(\alpha_1) = \begin{pmatrix} 0 & V_1 & & & \\ & 0 & V_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & V_t \\ & & & & 0 \end{pmatrix}.$$

One concludes that the classification problem of $L_t, t \geq 2$ is included in the one of L_2.

> **Step 3.** *If we could solve the classification problem for the quiver K_3 then we could solve it for any quiver Q.*

To prove the proposition concerning arbitrary quivers Q with vertex set Q_0 and set of arrows Q_1, one proceeds as follows:

$$Q_0 = \{1, \ldots, n\}, \quad Q_1 = \{\beta_1, \ldots, \beta_r\}, \quad \beta_j : s_j \to t_j.$$

Given a representation V of Q we define a representation $H(V)$ of L_t where we set $t = n + s$,

$$H(V)(0) = V(1) \oplus \cdots \oplus V(n)$$

and define $H(V)_i = H(V)(\beta_i), 1 \leq i \leq n$ to be the block matrix whose only non-zero block is $\mathbf{1}_{V(i)}$ at position (i, i) as well as $H(V)_{n+j} = H(V)(\beta_j), 1 \leq j \leq r$ the block matrix whose only non-zero block is $V(\beta_j)$ at position (t_i, s_i).

Then V is indecomposable if and only if $H(V)$ is indecomposable and two representations V, W of Q are isomorphic if and only if their images under H are isomorphic. This shows that the classification problem of Q is included in the classification problem of L_t.

4 The connection with the mutation class of a quiver

In the case where Q is a finite quiver without oriented cycles we can consider it's mutation class. If we denote by $\nu(Q)$ the supremum of the multiplicities of arrows occurring in all quivers mutation-equivalent to Q we get the following

Theorem 3. *i) Q is representation-finite $\Leftrightarrow$ $\nu(Q) = 1$*

ii) Q is tame $\Leftrightarrow$ $\nu(Q) = 2$

iii) Q is wild $\Leftrightarrow$ $\nu(Q) \geq 3$ $\Leftrightarrow$ $\nu(Q) = \infty$

iv) The mutation class of Q is finite iff Q has two vertices, is representation-finite or tame

Sketch of parts of the proof.

ii) This follows by the combination of i) and iii).

i) Recall that any quiver Q without loops or 2-cycles and with vertex set $\{1, \ldots, n\}$ corresponds to a $n \times n$ antisymmetric integer matrix B whose entry b_{ij} is the number of arrows $i \to j$ minus the number of arrows $j \to i$ in Q. The matrix B' corresponding to the mutation at k has then entries

$$b'_{ij} = \begin{cases} -b_{ji} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{else.} \end{cases}$$

It is shown in [5] that the inequality

$$|b_{xy}b_{yx}| \leq 3 \text{ for any } x, y \in \{1, \ldots, k\}$$

for the matrix indices in any mutation matrix is equivalent to the case where Q is an orientation of a Dynkin diagram, distinguishing possible mutations on sinks and sources of a quiver, on trees and on cycles, also coming along the equivalence with $\nu(Q) = 1$.

iii) The proof of this part is fairly involved and uses essentially concepts from homological algebra. We skip it here but refer to the example sketched in part two of this handout for an illustration. The interested reader is referred to [5] for a proof.

iv) Unfortunately, the reference given for a proof of this part is not freely available. □

5 Another prospect

As the category of representations $\mathrm{rep}_k(Q)$ of a quiver Q turns out to be equivalent to the category $\mathrm{mod}kQ$ of finitely-generated modules over the corresponding path algebra, the problem of classifying indecomposable representations of a quiver can be equally well posed as classifying the indecomposable modules of the path algebra. The original definition for k-algebras of wild representation type (e.g. the path algebra of a quiver Q) is the following:

> **Definition.** *A finite dimensional k-algebra A is of* wild representation type *if, for each finite dimensional k-algebra B, there exists a representation embedding functor $T : \mathrm{mod}B \to \mathrm{mod}A$.*

Thus, the classification of the indecomposable modules in $\mathrm{mod}A$ for A being representation-wild contains the classification of the indecomposable modules in $\mathrm{mod}B$. In particular, it can be shown to contain the classification of the finite indecomposable modules in $\mathrm{mod}k\langle t_1, \ldots, t_n\rangle$ of the free associative algebra $k\langle t_1, \ldots, t_n\rangle$ of polynomials in $n \geq 1$ non-commuting indeterminantes.

To connect this situation with the preceding example, consider the path algebra kK_3 of the extended Kronecker quiver K_3. It can be shown to be isomorphic to the algebra

$$B = \begin{pmatrix} k & 0 \\ k^3 & k \end{pmatrix}.$$

That B is representation-wild can be checked proving the validity of the following diagram

$$\mathrm{mod}k\langle t_1, t_2\rangle \xrightarrow{\simeq} \mathrm{rep}_k(L_2) \xrightarrow{R} \mathrm{rep}_k(K_3) \xrightarrow{\simeq} \mathrm{mod}B$$

where the path algebra kL_2 is isomorphic to the free associative algebra $k\langle t_1, t_2\rangle$ of polynomials in thwo non-commuting indeterminantes t_1, t_2 by the assignment $\varepsilon_1 \mapsto 1, \alpha_1 \mapsto t_1, \alpha_2 \mapsto t_2$.

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