Total positivity and cluster algebras

Reading group

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16. november 2010

Outline

1. Tagged triangulations
2. Cluster expansion formula for ordinary arcs
   - Tiles
   - The graph $G_{T^0,\gamma}$
   - Cluster expansion formula for ordinary arcs

Outline

(S, M) bordered surface with marked points

Bordered surface (S, M) is given by:

- S connected oriented 2-dimensional Riemann surface with boundary.
- M fixed nonempty set of **marked points** in the closure of S. Marked points in the interior of S are called **punctures**.

Restrictions

- *At least one marked point on each boundary component,*
- (S, M) *is not a sphere with one, two or three punctures,*
- (S, M) *is not a bigon,*
- (S, M) *is not a triangle without punctures.*

Tagged arcs

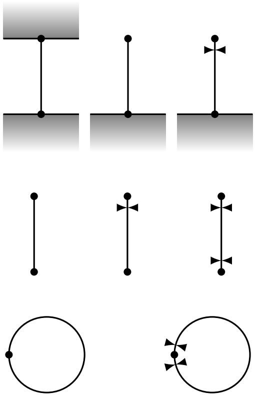
Definition

A **tagged arc** is an arc that does not cut out a once-punctured monogon and each of its ends is *tagged* **plain** or **notched** $\bowtie$, such that

- ends with endpoints lying on the boundary of S are tagged plain,
- both ends of a loop are tagged in the same way.

Tagged arcs

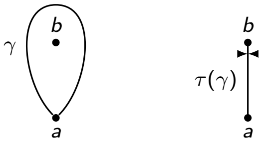
Example (Different types of tagged arcs)



Representing arcs by tagged arcs

γ arc in (S, M).
The **associated tagged** $\tau(\gamma)$ **arc** is given by

- if γ not cut out a once-punctured monogon:
  $\tau(\gamma) = \gamma$ with both ends tagged plain,
- if γ is a loop at marked point *a* cutting out a once-punctured monogon with puncture *b*:
  $\tau(\gamma) = \gamma'$ compatible with γ having endpoints a, b and with plain at *a* and notched at *b*.



Compatibility of tagged arcs

Definition

Tagged arcs γ, γ' are **compatible** if

- the untagged arcs γ^0, γ'^0 are compatible,
- $\gamma^0 = \gamma'^0$, then at least one end of γ is tagged in the same way as the corresponding end of γ',
- $\gamma^0 \neq \gamma'^0$ share an endpoint *a*, then both ends in *a* are tagged in the same way.

Remark

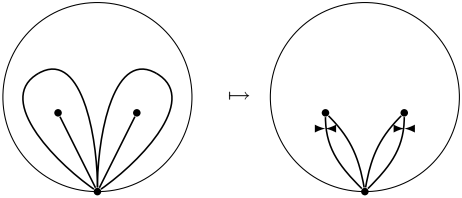
The map $\gamma \mapsto \tau(\gamma)$ *preserves compatibility.*

Tagged triangulations

Definition

A maximal collection of pairwise compatible tagged arcs is called **tagged triangulation**.

Example (Ideal triangulation and the corresponding tagged triangulation of a two-punctured monogon)



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Tile

(S, M) bordered surface with marked points.
γ a fixed arc, $T = \tau(T^0) =: (\tau_1, \tau_2, \ldots, \tau_n)$ a fixed tagged triangulation of (S, M), where $T^0 = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is an ideal triangulation of (S, M) not containing γ.

$s \in M$ starting point of γ, $t \in M$ endpoint of γ.
The d intersection points of γ and T^0 are

$$s = p_0, p_1, \ldots, p_{d+1} = t.$$

p_j is contained in $\gamma_{i_j} \in T^0, \quad (1 \leq j \leq d, 1 \leq i_j \leq n)$
Δ_{j-1} and Δ_j denote the two ideal triangles in T^0, which have side γ_{i_j}.

Tile

Definition

For each $p_j, 1 \leq j \leq d$, the **tile** G_j is defined to be the edge-labeled triangulated quadrilateral, obtained by the union of two edge-labeled triangles Δ_1^j, Δ_2^j glued at γ_{i_j}, which are given by

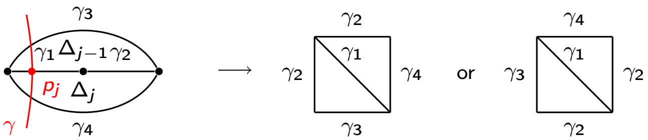
- if Δ_{j-1}, Δ_j are no self-folded triangles, then Δ_1^j, Δ_2^j corresponds to them and G_j is called **ordinary tile**.

Remark

The orientations of Δ_1^j, Δ_2^j agree or disagree with those of Δ_{j-1}, Δ_j, hence there are two possible planar embeddings of G_j.

Tile

Example

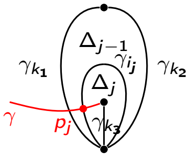


Tile

Definition

For each $p_j, 1 \leq j \leq d$, the **tile** G_j is defined to be the edge-labeled triangulated quadrilateral, obtained by the union of two edge-labeled triangles Δ_1^j, Δ_2^j glued at γ_{i_j}, which are given by

- if Δ_{j-1} or Δ_j is a self-folded triangle, then there are two cases
  1. γ intersect the loop γ_{i_j} and terminate at the puncture, then G_j is a ordinary tile given by Δ_1^j, Δ_2^j, which correspond to the triangles with sides $\{\gamma_{k_1}, \gamma_{k_2}, \gamma_{i_j} = \mathrm{l}(\gamma_{k_3})\}$ and $\{\gamma_{i_j} = \mathrm{l}(\gamma_{k_3}), \gamma_{k_3}, \gamma_{k_3}\}$.

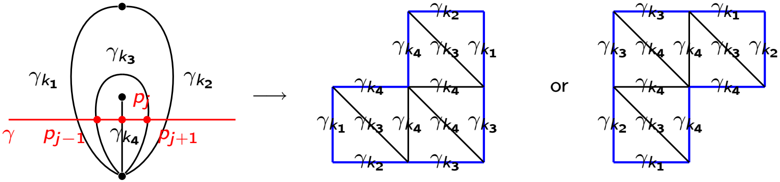


Tile

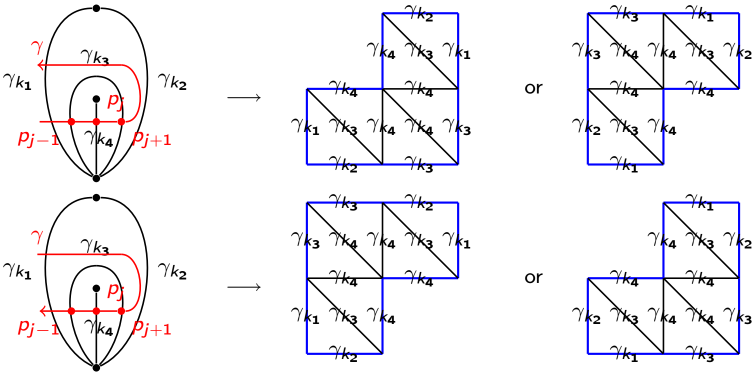
Definition

For each $p_j, 1 \leq j \leq d$, the **tile** G_j is defined to be the edge-labeled triangulated quadrilateral, obtained by the union of two edge-labeled triangles Δ_1^j, Δ_2^j glued at γ_{i_j}, which are given by

- if Δ_{j-1} or Δ_j is a self-folded triangle, then there are two cases
  2. γ intersect the loop, the folded side and the loop again, then to the three intersection points p_{j-1}, p_j, p_{j+1} it is associated a union of tales $G_{j-1} \cup G_j \cup G_{j+1}$, called **triple tile**.



Tile


γ
γ_{k_1}
γ_{k_2}
γ_{k_3}
γ_{k_4}
p_j
p_{j-1}
p_{j+1}
or
γ
γ_{k_1}
γ_{k_2}
γ_{k_3}
γ_{k_4}
p_j
p_{j-1}
p_{j+1}
or

Tile

Remark

In case

1. *there are again two possible planar embeddings of* G_j.
2. *there are two possible planar embeddings of the triple tile.*

Relative orientation

Definition

$\tilde{G}_j$ a planar embedding of an ordinary tile G_j ($1 \leq j \leq d$). The **relative orientaion** $\mathrm{rel}(\tilde{G}_j, T^0)$ of $\tilde{G}_j$ with respect to T^0 is given by

$$\mathrm{rel}(\tilde{G}_j, T^0) := \begin{cases} 1, & \text{if the triangles of } \tilde{G}_j \text{ agree in orientation} \\ & \text{with those of } T^0, \\ -1, & \text{if its triangles of } \tilde{G}_j \text{ disagree in orientation} \\ & \text{whith those of } T^0. \end{cases}$$

Remark

For a planar embedding of a triple tile $G_{j-1} \cup G_j \cup G_{j+1}$ is

$$\mathrm{rel}(\tilde{G}_{j-1}, T^0) = \mathrm{rel}(\tilde{G}_{j+1}, T^0).$$

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$\gamma_{[j]}$

The arcs $\gamma_{i_j}, \gamma_{i_{j+1}}$ $(1 \leq j \leq d-1)$ form two sides of the ideal triangle Δ_j in T^0.

$$\gamma_{[j]} := \begin{cases} \text{the third side in } \Delta_j, & \text{if } \Delta_j \text{is not self-folded,} \\ \text{the folded side in } \Delta_j, & \text{if } \Delta_j \text{ is self-folded.} \end{cases}$$

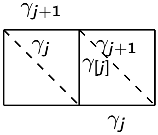
$G_{T^0,\gamma}$

Definition

$\overline{G}_{T^0,\gamma}$ is obtained from the tiles $G_1, G_2, \ldots, G_d$, by glueing them in the following way

- triple tiles stay glued together,
- for two adjacent ordinary tiles G_j and G_{j+1}, may be exterior tiles of triple tiles, G_{j+1} and $\tilde{G}_j$ are glued along the edge $\gamma_{[j]}$, choosing a planar embedding $\tilde{G}_{j+1}$ for G_{j+1} such that $\mathrm{rel}(\tilde{G}_{j+1}, T^0) \neq \mathrm{rel}(\tilde{G}_j, T^0)$.

$G_{T^0,\gamma}$ is the graph obtained from $\overline{G}_{T^0,\gamma}$ by removing the diagonal in each tile.



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Crossing Monomial, perfect matchings and weights

Definition

The **crossing monomial** of γ with respect to T^0 is

$$\operatorname{cross}(T^0, \gamma) := \prod_{j=1}^{d} x_{\gamma_{i_j}}.$$

Definition

A **perfect matching** of a Graph G is a subset P of the edges of G such that each vertex of G is incident to exactly one edge of P.

Definition

$\gamma_{j_1}, \ldots, \gamma_{j_r}$ are the edges of a perfect matching P of $G_{T^0,\gamma}$. The **weight** of P is

$$x(P) := \prod_{i=j_1}^{j_r} x_{\gamma_i}.$$

Perfect matchings

Remark

*$G_{T^0,\gamma}$ has exactly two perfect matchings, called **minimal matching** P_- and **maximal matching** P_+, which contain only boundary edges.*

Cluster expansion formula

Theorem

(S, M) bordered surface with marked points, $T^0 = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ an ideal triangulation, $T = (\tau_1, \tau_2, \ldots, \tau_n) = \tau(T^0)$ the associated tagged triangulation and $\mathcal{A}$ denotes the corresponding cluster algebra.
$\gamma \notin T^0$ an arc in (S, M) and $G_{T^0,\gamma}$ as before.
Then the Laurent expansion of x_γ with respect to T is given by

$$\frac{\sum_P x(P)}{\mathrm{cross}(T^0, \gamma)}$$

where the sum is over all perfect matchings P of $G_{T^0,\gamma}$.