Notizen zum Seminar

Wintersemester 2006/07

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1.1

Definition 1.1. *For a measurable function* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$*, we define its associated distribution function by*

$$\lambda_f(\alpha) = \mu(E_f^\alpha)\,,$$

where E_f^α *denotes the set* $\{x \in \mathbb{R}^n \;:\; |f(x)| > \alpha\}$ *with* $\alpha \geq 0$ *and* μ *the Lebesgue measure on* $\mathbb{R}^n$*.*

Lemma 1.2. *For a measurable function* f *and* $0 < p < \infty$*, we have*

$$\|f\|_{L^p}^p = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha)\, d\alpha\,. \tag{1.1}$$

Proof. From elementary calculus, we get

$$|f(x)|^p = p \int_0^{|f(x)|} \alpha^{p-1}\, d\alpha = p \int_0^\infty \alpha^{p-1} \chi_{\{\alpha < |f(x)|\}}\, d\alpha\,.$$

By integration over $\mathbb{R}^n$ and Fubini's theorem, it then follows

$$\|f\|_{L^p}^p = p \int_0^\infty \alpha^{p-1} \left(\int_{\mathbb{R}^n} \chi_{\{x\,:\,|f(x)|>\alpha\}}\, dx \right) d\alpha = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha)\, d\alpha\,.$$

□

Hardy-Littlewood Maximal Function

Definition 1.3. *For a locally integrable function* $f \in L^1_{loc}(\mathbb{R}^n)$*, we define its associated Hardy-Littlewood maximal function by*

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)|\, dy\,. \tag{1.2}$$

Theorem 1.4 (Hardy-Littlewood Maximal Theorem). *Let* $1 < p \le \infty$ *and* $f \in L^p(\mathbb{R}^n)$*. Then, we have*

$$\|Mf\|_{L^p} \le C\,\|f\|_{L^p}\,, \tag{1.3}$$

where the constant $C = C(n,p)$ *depends only on the dimension* n *and* p*. Moreover, for* $f \in L^1(\mathbb{R}^n)$ *and* $\alpha > 0$*, we have*

$$\mu(\{x : Mf(x) > \alpha\}) \le \frac{C}{\alpha}\,\|f\|_{L^1}\,, \tag{1.4}$$

where the constant $C = C(n)$ *depends only on* n*.*

Proof. In a first step we prove (1.4). – Let $E^\alpha_{Mf} = \{x : Mf(x) > \alpha\}$ denote the set where the Hardy-Littlewood maximal function of f is greater than $\alpha > 0$. For $x \in E^\alpha_{Mf}$, there exists by Definition 1.3 a ball $B_{r_x}(x)$ with radius $r_x > 0$ and center x, simply denoted by B^x, such that

$$\int_{B^x} |f(y)|\,dy > \alpha\,\mu(B^x)\,. \tag{1.5}$$

The family $\mathcal{F} = \{B^x \ : \ x \in E^\alpha_{Mf}\}$ of such balls clearly covers the set E^α_{Mf}. Using Lemma 1.5, we deduce the existence of a countable subfamily $\{B^{x_k}\}_{k\in\mathbb{N}}$ of disjoint balls in $\mathcal{F}$ satisfying

$$\sum_{k=1}^{\infty} \mu(B^{x_k}) \ge \frac{1}{5^n}\,\mu(E^\alpha_{Mf})\,.$$

Applying (1.5) to each of these disjoint balls, we then obtain

$$\|f\|_{L^1} \ge \int_{\bigcup_{k=1}^\infty B^{x_k}} |f(y)|\,dy > \sum_{k=1}^{\infty} \alpha\,\mu(B^{x_k}) \ge \frac{\alpha}{5^n}\,\mu(E^\alpha_{Mf})\,.$$

This shows (1.4) with $C = 5^n$.

In a second step, we want to show (1.3). – Since the case $p = \infty$ is trivial with $C(n,\infty) = 1$, we assume that $1 < p < \infty$. For $\alpha > 0$, let

$$f_1(x) = \begin{cases} f(x) & \text{if } |f(x)| \ge \alpha/2 \\ 0 & \text{if } |f(x)| < \alpha/2\,. \end{cases}$$

Then, we have $|f(x)| \le |f_1(x)| + \alpha/2$ and also $|Mf(x)| \le |Mf_1(x)| + \alpha/2$, for all $x \in \mathbb{R}^n$. Therefore, we get

$$E^\alpha_{Mf} = \{x : Mf(x) > \alpha\} \subset \{x : Mf_1(x) > \alpha/2\} = E^{\alpha/2}_{Mf_1}\,.$$

Since $f_1 \in L^1(\mathbb{R}^n)$, we can apply (1.4) to f_1 in order to get

$$\mu(E^{\alpha/2}_{Mf_1}) \le \frac{2C}{\alpha}\,\|f_1\|_{L^1}\,.$$

Thus, we arrive at

$$\mu(E_{Mf}^{\alpha}) \leq \mu(E_{Mf_1}^{\alpha/2}) \leq \frac{2C}{\alpha}\,\|f_1\|_{L^1} \leq \frac{2C}{\alpha}\int_{\{x\,:\,|f(x)|\geq\alpha/2\}} |f(x)|\,dx\,. \tag{1.6}$$

Next, we deduce from Lemma 1.2 that

$$\begin{aligned}
\|Mf\|_{L^p}^p &= p\int_0^\infty \alpha^{p-1}\lambda_{Mf}(\alpha)\,d\alpha \\
&\overset{(1.6)}{\leq} p\int_0^\infty \alpha^{p-1}\left(\frac{2C}{\alpha}\int_{\{x\,:\,|f(x)|\geq\alpha/2\}}|f(x)|\,dx\right)d\alpha \\
&= p\int_0^\infty \alpha^{p-1}\left(\frac{2C}{\alpha}\int_{\mathbb{R}^n}\chi_{\{x\,:\,|f(x)|\geq\alpha/2\}}|f(x)|\,dx\right)d\alpha\,.
\end{aligned}$$

Using Fubini's theorem as in the proof of Lemma 1.84, it follows

$$\begin{aligned}
\|Mf\|_{L^p}^p &\leq 2C\,p\int_{\mathbb{R}^n}|f(x)|\left(\int_0^{2|f(x)|}\frac{\alpha^{p-1}}{\alpha}\,d\alpha\right)dx \\
&= \frac{2C\,p}{p-1}\int_{\mathbb{R}^n}|f(x)|\,2^{p-1}|f(x)|^{p-1}\,dx\,,
\end{aligned}$$

since $p > 1$ by assumption. Thus we arrive at the desired result

$$\|Mf\|_{L^p} \leq \left(\frac{2^pC\,p}{p-1}\right)^{1/p}\|f\|_{L^p}\,.$$

□

Lemma 1.5 (Vitali-type Covering Lemma). *Let $E \subset \mathbb{R}^n$ be measurable and suppose that $E \subset \bigcup_j B_j$, where the family $\{B_j\}_{j\in J}$ is contained of balls with bounded diameter, i.e., $\sup_j diam(B_j) = C < \infty$. Then, there exists a countable disjoint subfamily $\{B_{j_k}\}_{k\in\mathbb{N}}$ such that*

$$\mu(E) \leq 5^n\sum_{k=1}^{\infty}\mu(B_{j_k})\,. \tag{1.7}$$

The Critical Case $p = 1$

We want to emphasize that taking the Hardy-Littlewood maximal function is not a bounded operation on $L^1(\mathbb{R}^n)$. This can be directly deduced from the following observation: If $f \in L^1(\mathbb{R}^n)$ and $f \not\equiv 0$, then Mf is *not* in $L^1(\mathbb{R}^n)$. To see this, let $\varepsilon > 0$ small enough and because f vanishes not identically on $\mathbb{R}^n$, there exists $r_0 > 0$ such that

$$\int_{B_{r_0}}|f(x)|\,dx \geq \varepsilon\,.$$

Note that for $|x| > r_0$, we have $B_{r_0} \subset B_{2|x|}(x)$. Thus, it follows

$$\begin{aligned} Mf(x) &= \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)|\,dy \\ &\geq \frac{1}{\mu(B_{2|x|}(x))} \int_{B_{2|x|}(x)} |f(y)|\,dy \\ &\geq \frac{1}{\mu(B_{2|x|}(x))} \int_{B_{r_0}} |f(y)|\,dy \geq \frac{C\,\varepsilon}{|x|^n}\,, \end{aligned}$$

showing that the integrability of Mf fails at infinity.

Moreover, even if we restrict our attention to bounded subsets of $\mathbb{R}^n$ the requirement of f (local) integrable is not sufficient for the (local) integrability of Mf. We illustrate this fact by the following example: For $n = 1$ consider the positive function

$$f(t) = \frac{1}{t(\log t)^2} \chi_{(0,1)}\,,$$

which is integrable on $[0, 1/2]$[1]. For $t \in (0, 1/2)$, let $B_{2t}(t) = (0, 2t)$ and we have

$$\begin{aligned} Mf(t) &\geq \frac{1}{2t} \int_0^{2t} \frac{1}{t(\log t)^2}\,dt \\ &= \frac{1}{2t} \left(-\frac{1}{\log t} \right) \Big|_0^{2t} = -\frac{1}{2t(\log 2t)}\,. \end{aligned}$$

This directly gives that Mf is not integrable over the interval $[0, 1/2]$.

The next proposition, however, shows that if we impose stronger conditions on f then the local integrability of the Hardy-Littlewood maximal function Mf can be deduced.

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[1] More generally, for $\alpha > 0$, we consider the following functions on $\mathbb{R}^n$:

$$f(x) = \frac{1}{\|x\|^n \, \log(1/\|x\|)^{1+\alpha}} \chi_{B_1} \leq \frac{1}{\|x\|^n \, |\log \|x\|\,|^{1+\alpha}} \chi_{B_1}\,.$$

Integration over $B_{1/2}$ in polar coordinates gives

$$\int_{B_{1/2}} f(x)\,dx \leq C \int_0^{1/2} \frac{r^{n-1}}{r^n \, |\log r|^{1+\alpha}}\,dr$$

Introducing the new variable $s = |\log r|$, we obtain

$$\int_{B_{1/2}} f(x)\,dx \leq \int_{|\log(1/2)|}^{\infty} \frac{1}{s^{1+\alpha}}\,ds\,.$$

Since $1/(1+\alpha) < 1$, we deduce that $f \in L^1(B_{1/2})$.

Proposition 1.6. *Let B be a bounded subset of $\mathbb{R}^n$ and assume that $f \in L\log L$, i.e.,*

$$\int_{\mathbb{R}^n} |f(x)| \log^+ |f(x)|\, dx < \infty\,,$$

where $\log^+ |f(x)| = \max\{0, \log |f(x)|\}$. *Then we have that $Mf \in L^1(B)$.*

Proof. From (1.1), we directly deduce that

$$\|Mf\|_{L^1(B)} \leq 2 \int_0^\infty \lambda_{Mf}(2\alpha)\, d\alpha\,,$$

and hence

$$\|Mf\|_{L^1(B)} \leq 2\,\mu(B) + 2 \int_1^\infty \lambda_{Mf}(2\alpha)\, d\alpha\,. \tag{1.8}$$

Proceeding as in the second step of the proof for Theorem 1.4, we obtain

$$\begin{aligned} \int_1^\infty \lambda_{Mf}(2\alpha)\, d\alpha &\overset{(1.6)}{\leq} \int_1^\infty \left(\frac{C}{\alpha} \int_{\mathbb{R}^n} \chi_{\{x\,:\,|f(x)| \geq \alpha\}} |f(x)|\, dx \right) d\alpha \\ &= C \int_{\mathbb{R}^n} |f(x)| \left(\int_1^{\max\{1, |f(x)|\}} \frac{1}{\alpha}\, d\alpha \right) dx\,. \end{aligned}$$

A straightforward integration yields

$$\int_1^\infty \lambda_{Mf}(2\alpha)\, d\alpha \leq C \int_{\mathbb{R}^n} |f(x)| \log^+ |f(x)|\, dx\,.$$

Inserting this in (1.8), we arrive at

$$\|Mf\|_{L^1(B)} \leq 2\,\mu(B) + 2C \int_{\mathbb{R}^n} |f(x)| \log^+ |f(x)|\, dx\,, \tag{1.9}$$

where the right-hand side is finite by assumption. □

The Calderón-Zygmund Decomposition

Theorem 1.7 (Calderón-Zygmund Decomposition). *Let $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$ and let $\alpha > 0$. Then there exists a sequence of disjoint cubes $(C_k)_{k \in \mathbb{N}}$ such that*

(i) The average of f on all cubes is bounded from below and above by

$$\alpha < \frac{1}{\mu(C_k)} \int_{C_k} f(x)\, dx \leq 2^n \alpha\,. \tag{1.10}$$

(ii) On the complement Ω^c of the union $\Omega = \bigcup_{k=1}^\infty C_k$, we have

$$f(x) \leq \alpha \qquad a.e.\,. \tag{1.11}$$

(iii) There exists a constant $C = C(n)$ depending only on the dimension n such that

$$\mu(\Omega) \leq \frac{C}{\alpha} \|f\|_{L^1} \,. \tag{1.12}$$

Proof. We divide $\mathbb{R}^n$ into a mesh of equal cubes chosen large enough such that their volume is larger or equal than $\|f\|_{L_1}/\alpha$. Thus, for every cube C_0 in this mesh, we have

$$\frac{1}{\mu(C_0)} \int_{C_0} f(x)\, dx \leq \alpha \,. \tag{1.13}$$

Next, we fix a cube C_0 in the initial mesh. We decompose it into 2^n equal disjoint cubes with half of the side-length. For the resulting cubes, there are now two possibilities: Either (1.13) still holds or (1.13) is violated. Cubes of the first case are the *good* cubes, denoted by $\mathcal{C}_1^g$, and the *bad* cubes of the second case are denoted by $\mathcal{C}_1^b$. In a next step, we decompose all cubes $\mathcal{C}_1^g$ into equal disjoint cubes with half side-length and leave the cubes $\mathcal{C}_1^b$ unchanged. The resulting cubes for which an estimate of the form (1.13) still holds are denoted by $\mathcal{C}_2^g$ and the remaining ones by $\mathcal{C}_2^b$. Then, we proceed as before dividing the cubes $\mathcal{C}_2^g$ and leaving the cubes $\mathcal{C}_2^b$ unchanged. – Repeating this procedure for each cube in the initial mesh, we can define $\Omega = \bigcup_{k=1}^{\infty} C_k$ as the union of all cubes which violate in some step of the decomposition process an estimate of the form (1.13). (These are precisely those cubes with an upper index b for *bad*.)

Note that for a cube C_i^b in $\mathcal{C}_i^b$ obtained in the i-th step, we have

$$\frac{1}{\mu(C_i^b)} \int_{C_i^b} f(x)\, dx > \alpha \,. \tag{1.14}$$

Since $2^n \mu(C_i^b) = \mu(C_{i-1}^g)$, where C_{i-1}^g is any cube in $\mathcal{C}_{i-1}^g$, we then deduce

$$\alpha < \frac{1}{\mu(C_i^b)} \int_{C_i^b} f(x)\, dx \leq \frac{2^n}{\mu(C_{i-1}^g)} \int_{C_{i-1}^g} f(x)\, dx \leq 2^n \alpha \,.$$

This shows (i) of the theorem.

In order to show (ii), we note that by a variant of Lebesgue's differentiation theorem (see) almost everywhere

$$f(x) = \lim_{d \to 0} \frac{1}{\mu(C_{x,d})} \int_{C_{x,d}} f(y)\, dy \,,$$

where $C_{x,d}$ denotes a cube containing $x \in \mathbb{R}^n$ with diameter d. By construction of the decomposition, there exists for every $x \in \Omega^c$ a diameter $d_x > 0$ such that all cubes $C_{x,d}$ with diameter $d < d_0$ satisfy an estimate of the form (1.13). This implies directly that $f(x) \leq \alpha$ for a.e. $x \in \Omega^c$.

The last part (iii) of the theorem can be established as follows:

$$\mu(\Omega) = \sum_{k=1}^{\infty} \mu(C_k) \overset{(1.15)}{<} \frac{1}{\alpha} \int_{\Omega} f(x)\, dx \leq \frac{1}{\alpha} \|f\|_{L^1} .$$

□

Definition 1.8. *Let $1 \leq p, q \leq \infty$ and let T be a mapping from $L^p(\mathbb{R}^n)$ to the space of measurable functions. For $1 \leq q \leq \infty$, we say that the mapping T is of strong type (p, q) – or simply of type (p, q) – if*

$$\|Tf\|_{L^q} \leq C \|f\|_{L^p} ,$$

where the constant C is independent of $f \in L^p(\mathbb{R}^n)$. For the case of $q < \infty$, we say that T is of weak type (p, q) if

$$\mu(\{x \in \mathbb{R}^n \ : \ |Tf(x)| > \alpha\}) \leq C \left(\frac{1}{\alpha} \|f\|_{L^p}\right)^q ,$$

where the constant C is independent of f and $\alpha > 0$. For $q = \infty$, we say that T is of weak type (p, ∞) if T is of type (p, ∞).

Remark. For $q < \infty$, we have by Chebyshev's inequality

$$\alpha^q \, \mu(\{x \, : \, |Tf(x)| > \alpha\}) \leq \|Tf\|_{L^q}^q \leq (C \|f\|_{L^p})^q ,$$

implying that T being of type (p, q) is also of weak type (p, q).

We also define $L^{p_1} + L^{p_2}(\mathbb{R}^n)$ as the space of all functions f which can be written as $f = f_1 + f_2$ with $f_1 \in L^{p_1}(\mathbb{R}^n)$ and $f_2 \in L^{p_2}(\mathbb{R}^n)$. By splitting a function in its small and large parts, one can show that $L^p(\mathbb{R}^n) \subset L^{p_1} + L^{p_2}(\mathbb{R}^n)$, for $p_1 \leq p \leq p_2$ with $p_1 < p_2$.

Theorem 1.9 (Marcinkiewicz Interpolation Theorem). *Let $1 < r \leq \infty$ and suppose that T is a sublinear operator from $L^1 + L^r(\mathbb{R}^n)$ to the space of measurable functions, i.e., for all $f, g \in L^1 + L^r(\mathbb{R}^n)$, the following pointwise estimate holds:*

$$|T(f + g)| \leq |Tf| + |Tg| . \tag{1.15}$$

Moreover, assume that T is of weak type $(1, 1)$ and also of weak type (r, r). Then, for $1 < p < r$, we have that T is of type (p, p) meaning that

$$\|Tf\|_{L^p} \leq C \|f\|_{L^p} ,$$

for all $f \in L^p(\mathbb{R}^n)$.

Remark. Because of the last theorem and the fact that the Hardy-Littlewood maximal function is sublinear, we can directly deduce (1.3) in Theorem 1.4 from (1.4) – saying that the operator M is of weak type $(1, 1)$ – and the obvious observation that M is of type (∞, ∞).

Theorem 1.10. *Let $K \in L^2(\mathbb{R}^n)$ and assume the following:*

(i) The Fourier transform $\hat{K}$ of K is essentially bounded by the constant A, i.e.,

$$\|\hat{K}\|_{L^\infty} \leq A\,. \tag{1.16}$$

(ii) The function K satisfies the so-called Hörmander condition

$$\int_{2\|y\|\leq\|x\|} \big|K(x-y) - K(x)\big|\, dx \leq B\,, \qquad \text{for } \|y\| > 0\,. \tag{1.17}$$

Moreover, let T be the well-defined convolution operator on $L^1(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$, with $1 < p < \infty$, given pointwise by

$$Tf(x) = K \star f(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\, dy\,. \tag{1.18}$$

Then, there exists a constant $C = C(n,p,A,B)$ – but independent of the L^2-norm of K – such that

$$\|Tf\|_{L^p} \leq C\, \|f\|_{L^p}\,. \tag{1.19}$$

Remark. a) In the previous theorem, the kernel K is assumed to be in $L^2(\mathbb{R}^n)$ in order to make the convolution operator T well defined on $L^1(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$, for $1 < p < \infty$. In fact, by Young's inequality for convolutions we have

$$\|Tf\|_{L^2} \leq \|K\|_{L^2}\|f\|_{L^1}\,,$$

where we are explicitly using the fact that $f \in L^1(\mathbb{R}^n)$.

b) Note that T is a densely defined *linear* operator on $L^p(\mathbb{R}^n)$. More precisely, the operator is well-defined on the dense linear subset $L^1(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$. Later, from (1.19), we can deduce that T can be extended to *all* of $L^p(\mathbb{R}^n)$ by continuity.

Proof. The proof is divided in the following three steps: First, we show that the convolution operator T is of weak type $(2,2)$. In a second step, we establish that T is of weak type $(1,1)$, which is the most difficult part of the proof. Finally we obtain the result (1.19) by Marcinkiewicz's interpolation theorem and a density argument.

First step: Let $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, then for the Fourier transform $\widehat{Tf}$ of $Tf \in L^2(\mathbb{R}^n)$, we have

$$\|\widehat{Tf}\|_{L^2} = \|\widehat{K\star f}\|_{L^2} = \|\hat{K}\,\hat{f}\|_{L^2} \overset{(1.16)}{\leq} A\,\|f\|_{L^2}\,.$$

Since $\|\widehat{Tf}\|_{L^2} = \|Tf\|_{L^2}$ by Plancherel's theorem, we then obtain

$$\|Tf\|_{L^2} \leq A\,\|f\|_{L^2}\,. \tag{1.20}$$

This shows that T is of type $(2,2)$, which also implies that T is of weak type $(2,2)$, i.e.,

$$\mu(\{x\,:\,|Tf(x)|>\alpha\}) \leq \frac{A^2}{\alpha^2}\,\|f\|_{L^2}^2\,. \tag{1.21}$$

Second step: Let $\alpha > 0$. Then we apply the Calderón-Zygmund Decomposition 1.7 to $0 \leq |f| \in L^1(\mathbb{R}^n)$ and α. The resulting countable family of disjoint cubes will be denoted by $\{C_k\}_{k\in\mathbb{N}}$ and we write $\Omega = \bigcup_{k=1}^{\infty} C_k$ for their union.

Now, we define

$$g(x) = \begin{cases} f(x) & \text{for } x \in \Omega^c \\ \dfrac{1}{\mu(C_k)} \displaystyle\int_{C_k} f(y)\,dy & \text{for } x \in C_k\,. \end{cases} \tag{1.22}$$

Writing f as sum of a *good* and a *bad* function, namely $f = g + b$, it follows that b has the following form:

$$b = \sum_{k=1}^{\infty} b_k\,, \tag{1.23}$$

with

$$b_k(x) = \left(f(x) - \frac{1}{\mu(C_k)}\int_{C_k} f(y)\,dy\right)\chi_{C_k}(x)\,.$$

Since by definition of the convolution operator T

$$|Tf(x)| \leq |Tg(x)| + |Tb(x)|\,, \tag{1.24}$$

for all $x \in \mathbb{R}^n$, we get

$$\begin{aligned}\mu(\{x\,:\,|Tf(x)|>\alpha\}) \leq\ & \mu(\{x\,:\,|Tg(x)|>\alpha/2\}) \\ & +\mu(\{x\,:\,|Tb(x)|>\alpha/2\})\,.\end{aligned} \tag{1.25}$$

In order to get an estimate for the first term on the right-hand side of (1.25), we first claim that g is an element of $L^2(\mathbb{R}^n)$. – From $|g(x)| \leq \alpha$ for $x \in \Omega^c$, we get

$$\begin{aligned}\|g\|_{L^2}^2 &= \int_{\Omega^c} |g(x)|^2\,dx + \int_{\Omega} |g(x)|^2\,dx \\ &= \int_{\Omega^c} \alpha|g(x)|\,dx + \int_{\Omega} |g(x)|^2\,dx\,,\end{aligned} \tag{1.26}$$

and the second term on the right-hand side can be bounded due to (1.10) by

$$\begin{aligned}\int_{\Omega} |g(x)|^2\,dx &= \sum_{k=1}^{\infty}\int_{C_k} |g(x)|^2\,dx \\ &\overset{(1.22)}{\leq} \sum_{k=1}^{\infty}\int_{C_k}\left(\frac{1}{\mu(C_k)}\int_{C_k}|f(x)|\,dx\right)^2 dx \\ &\leq \sum_{k=1}^{\infty}\int_{C_k}(2^n\alpha)^2\,dx = C\,\alpha^2\mu(\Omega)\,. \qquad (1.27)\end{aligned}$$

Inserting this into (1.26) and using also (1.12), we arrive at

$$\begin{aligned}\|g\|_{L^2}^2 &\leq \alpha\,\|f\|_{L^1} + C\,\alpha^2\mu(\Omega) \\ &\leq \alpha\,\|f\|_{L^1} + C\,\alpha\,\|f\|_{L^1} \leq (C+1)\alpha\,\|f\|_{L^1}\,,\end{aligned}$$

showing the claim. As a consequence, we can apply (1.21) to $g \in L^2(\mathbb{R}^n)$ in order to get the following estimate for the first term on the right-hand side of (1.25):

$$\begin{aligned}\mu(\{x\,:\,|Tg(x)| > \alpha/2\}) &\leq \frac{C}{\alpha^2}\|g\|_{L^2}^2 \\ &\leq \frac{C}{\alpha}\|f\|_{L^1}\,. \qquad (1.28)\end{aligned}$$

Next, we want to obtain an estimate for the second term on the right hand-side of (1.25). – For this purpose, we expand each cube C_k in the Calderón-Zygmund decomposition by the factor $2\sqrt{n}$ leaving its center c_k fixed. The new bigger cubes are denoted by $\tilde{C}_k$ and its union by $\tilde{\Omega} = \bigcup_{k=1}^{\infty}\tilde{C}_k$. It is easy to see that $\Omega \subset \tilde{\Omega}$, $\tilde{\Omega}^c \subset \Omega^c$ and $\mu(\tilde{\Omega}) \leq (2\sqrt{n})^n\,\mu(\Omega)$. Moreover, for $x \notin \tilde{C}_k$, we have

$$\|x - c_k\| \geq 2\,\|y - c_k\|\,, \qquad \text{for all } \ y \in C_k\,. \qquad (1.29)$$

Now, let c_k denote the center of the cube C_k. Then, we can write

$$\begin{aligned}Tb(x) = \sum_{k=1}^{\infty} Tb_k(x) &= \sum_{k=1}^{\infty}\int_{C_k} K(x-y)b_k(y)\,dy \\ &= \sum_{k=1}^{\infty}\int_{C_k}\big(K(x-y) - K(x-c_k)\big)b_k(y)\,dy\,,\end{aligned}$$

being a direct consequence of the fact that for all C_k

$$\int_{C_k} b_k(y)\,dy = \int_{C_k}\left(f(y) - \frac{1}{\mu(C_k)}\int_{C_k} f(z)\,dz\right)dy = 0\,.$$

This then leads to

$$\begin{aligned}\int_{\tilde{\Omega}^c} |Tb(x)|\,dx &\le \sum_{k=1}^{\infty} \int_{\tilde{\Omega}^c} \left(\int_{C_k} \big|K(x-y) - K(x-c_k)\big|\,\big|b_k(y)\big|\,dy \right) dx \\ &\le \sum_{k=1}^{\infty} \int_{\tilde{C}_k^c} \left(\int_{C_k} \big|K(x-y) - K(x-c_k)\big|\,\big|b_k(y)\big|\,dy \right) dx \\ &= \sum_{k=1}^{\infty} \int_{C_k} \left(\int_{\tilde{C}_k^c} \big|K(x-y) - K(x-c_k)\big|\,dx \right) \big|b_k(y)\big|\,dy\,. \end{aligned}$$

Setting $\bar{x} = x - c_k$, $\bar{y} = y - c_k$ and using (1.29), the integral in parenthesis becomes

$$\int_{\tilde{C}_k^c} \big|K(x-y) - K(x-c_k)\big|\,dx \le \int_{2\|\bar{y}\| \le \|\bar{x}\|} \big|K(\bar{x}-\bar{y}) - K(\bar{x})\big|\,d\bar{x}\,.$$

The assumption (1.17) of the theorem, then implies that

$$\int_{\tilde{\Omega}^c} |Tb(x)|\,dx \le B \sum_{k=1}^{\infty} \int_{C_k} |b_k(y)|\,dy \le C\,\|f\|_{L^1}\,. \tag{1.30}$$

At this stage, we are ready to give the following estimate for the second term in (1.25):

$$\begin{aligned}\mu(\{x \in \mathbb{R}^n \,:\, |Tb(x)| > \alpha/2\}) &\le \mu(\{x \in \tilde{\Omega}^c \,:\, |Tb(x)| > \alpha/2\}) + \mu(\tilde{\Omega}) \\ &\overset{(1.30)}{\le} \frac{2C}{\alpha}\,\|f\|_{L^1} + (2\sqrt{n})^n\,\mu(\Omega) \\ &\overset{(1.12)}{\le} \frac{2C}{\alpha}\,\|f\|_{L^1} + \frac{C}{\alpha}\,\|f\|_{L^1} \le \frac{C}{\alpha}\,\|f\|_{L^1}\,. \end{aligned} \tag{1.31}$$

Combining (1.28) with (1.31), we end up with

$$\mu(\{x \,:\, |Tf(x)| > \alpha\}) \le \frac{C}{\alpha}\,\|f\|_{L^1}\,, \tag{1.32}$$

showing that the convolution operator T is of weak type $(1,1)$.

Third step: Note that we have already shown the inequality (1.19) in the case of $p = 2$ in (1.20). – Putting $r = 2$ in Marcinkiewicz Interpolation Theorem 1.9 and using the fact that T is of weak type $(1,1)$, respectively $(2,2)$, by (1.21), respectively (1.32), we conclude that

$$\|Tf\|_{L^p} \le C\,\|f\|_{L^p}\,, \tag{1.33}$$

for $1 < p < 2$.

For the case $2 < p < \infty$, we will use a *duality argument*. – Consider the dual space $L^q(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$ with $1/p + 1/q = 1$. We easily see that

$1 < q < 2$. Now, let $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, then the L^p-norm of Tf is given by the following expression:

$$\|Tf\|_{L^p} = \sup_{\substack{g \in L^1 \cap L^q \\ \|g\|_{L^q} \leq 1}} \left| \int_{\mathbb{R}^n} Tf(x) g(x)\, dx \right| . \tag{1.34}$$

We calculate

$$\begin{aligned} \left| \int_{\mathbb{R}^n} Tf(x) g(x)\, dx \right| &= \left| \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x-y) f(y)\, dy \right) g(x)\, dx \right| \\ &= \left| \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x-y) g(x)\, dx \right) f(y)\, dy \right| , \end{aligned}$$

where Fubini's theorem was applied because of $K \in L^2(\mathbb{R}^n)$ and the assumptions on g and f. For the first integral, we conclude[2] from (1.33) that it is an element of $L^p(\mathbb{R}^n)$. Using Hölder's inequality, we end up with

$$\begin{aligned} \sup_{\substack{g \in L^1 \cap L^q \\ \|g\|_{L^q} \leq 1}} \left| \int_{\mathbb{R}^n} Tf(x) g(x)\, dx \right| &\leq \int_{\mathbb{R}^n} \left| \left(\int_{\mathbb{R}^n} K(x-y) g(x)\, dx \right) f(y) \right| dy \\ &\overset{(1.33)}{\leq} C \|g\|_{L^q} \|f\|_{L^p} \leq C \|f\|_{L^p} . \end{aligned}$$

This establishes the theorem. □

Singular Integral Operators II

We generalize Theorem 1.10 in the sense that now the L^2-boundedness of the convolution operator T follows from conditions imposed on the kernel K and not directly from the assumptions.

Theorem 1.11. *Let $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function such that*

$$|K(x)| \leq \frac{A}{\|x\|^n} , \qquad \textit{for } \|x\| > 0 . \tag{1.35a}$$

$$\int_{2\|y\| \leq \|x\|} \big| K(x-y) - K(x) \big|\, dx \leq B , \qquad \textit{for } \|y\| > 0 . \tag{1.35b}$$

$$\int_{R_1 < \|x\| < R_2} K(x)\, dx = 0 , \qquad \textit{for } 0 < R_1 < R_2 < \infty . \tag{1.35c}$$

For $\varepsilon > 0$ and $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$, we set

$$T_\varepsilon f(x) = \int_{\|y\| \geq \varepsilon} f(x-y) K(y)\, dy . \tag{1.36}$$

[2] That the kernel $K(x)$ is replaced by $K(-x)$ has no significance.

Then, we have

$$\|T_\varepsilon f\|_{L^p} \leq C\,\|f\|_{L^p}\,, \tag{1.37}$$

where the constant C *is independent of* ε *and* f*. Moreover, there exists* $Tf \in L^p(\mathbb{R}^n)$ *such that*

$$T_\varepsilon f \longrightarrow Tf \quad \text{in } L^p \qquad (\varepsilon \longrightarrow 0)\,, \tag{1.38}$$

for all $f \in L^p(\mathbb{R}^n)$*.*

Remark. The singular integral defined in (1.36) is absolutely convergent. To see this, note that due to (1.35a) we have that $K \in L^{p'}(\mathbb{R}^n \setminus B_\varepsilon)$, where $1 < p'$ is the Hölder conjugate exponent of p. From Young's inequality, it then follows that $\|T_\varepsilon f\|_\infty \leq \|f\|_{L^p}\|K\|_{L^{p'}}$.

Proof. For every $\varepsilon > 0$, we define

$$K_\varepsilon(x) = \begin{cases} K(x) & \text{if } \|x\| \geq \varepsilon \\ 0 & \text{if } \|x\| < \varepsilon\,. \end{cases} \tag{1.39}$$

We observe that $K_\varepsilon \in L^2(\mathbb{R}^n)$ and that the Hörmander condition (1.35b) also holds for K_ε. Moreover, we will show in Appendix 1.84 that

$$\|\hat{K}_\varepsilon\|_\infty \leq C\,, \tag{1.40}$$

where the constant $C = C(n)$ only depends on the dimension n and *not* on ε. Applying Theorem 1.10 to the kernels K_ε, $\varepsilon > 0$, we obtain (1.37) as direct consequence, since

$$T_\varepsilon f(x) = \int_{\mathbb{R}^n} f(x-y)K_\varepsilon(y)\,dy\,.$$

In a next step, we fix a function $f \in C_c^1(\mathbb{R}^n)$ and write

$$\begin{aligned} T_\varepsilon f(x) &= \int_{1\leq\|y\|} f(x-y)K(y)\,dy + \int_{\varepsilon\leq\|y\|\leq 1} f(x-y)K(y)\,dy \\ &= \int_{\mathbb{R}^n} f(x-y)K_1(y)\,dy + \int_{\varepsilon\leq\|y\|\leq 1} \big(f(x-y)-f(x)\big)K(y)\,dy\,. \end{aligned} \tag{1.41}$$

Note that the cancellation property (1.35c) is used for the second term on the right-hand side. Because of the regularity assumptions on f, we can apply the mean value theorem in order to get the existence of a constant C such that

$$\big|\big(f(x-y)-f(x)\big)K(y)\big| \leq C\,\|y\|\,|K(y)| \overset{(1.35a)}{\leq} \frac{C\,A}{\|y\|^{n-1}}\,, \tag{1.42}$$

for all $x \in \mathbb{R}^n$. This being integrable for $y \in B_1 \subset \mathbb{R}^n$, we deduce by dominated convergence theorem for the second integral on the right-hand side of (1.41) that

$$\lim_{\varepsilon \to 0} \int_{\varepsilon \le \|y\| \le 1} \big(f(x-y) - f(x)\big) K(y)\, dy = \int_{\|y\| \le 1} \big(f(x-y) - f(x)\big) K(y)\, dy\,, \tag{1.43}$$

for all $x \in \mathbb{R}^n$. At this stage, we can define

$$Tf(x) := \lim_{\varepsilon \to 0} T_\varepsilon f(x) = \int_{\mathbb{R}^n} f(x-y) K(y)\, dy\,, \tag{1.44}$$

for all $x \in \mathbb{R}^n$ and $f \in C_c^1(\mathbb{R}^n)$.

Writing (1.43) as $\lim_{\varepsilon \to 0} g_\varepsilon(x) = g(x)$, we directly deduce that

$$\big|g_\varepsilon(x) - g(x)\big|^p \longrightarrow 0 \qquad (\varepsilon \longrightarrow 0)\,,$$

for all $x \in \mathbb{R}^n$. Consider now the compact set $S = \{x \in \mathbb{R}^n \,:\, \mathrm{dist}(K, x) \le 1\}$ with K the support of $f \in C_c^1(\mathbb{R}^n)$, we obtain[3]

$$\begin{aligned} |g_\varepsilon(x)| &\le \chi_S(x) \int_{\varepsilon \le \|y\| \le 1} \big|f(x-y) - f(x)\big| |K(y)|\, dy \\ &\overset{(1.42)}{\le} CA\, \chi_S(x) \int_{B_1} \frac{1}{\|y\|^{n-1}}\, dy \le C\chi_S(x)\,. \end{aligned}$$

The right-hand side being independent of ε and integrable over $\mathbb{R}^n$, we conclude that

$$\big|g_\varepsilon(x) - g(x)\big|^p \le C\left(|g_\varepsilon(x)|^p + |g(x)|^p\right)$$

is still integrable. Thus, we can apply dominated convergence to arrive at

$$\int_{\mathbb{R}^n} \big|g_\varepsilon(x) - g(x)\big|^p\, dx \longrightarrow 0 \qquad (\varepsilon \longrightarrow 0)\,.$$

On the other hand, the first integral in (1.41) is an L^p-function for $p > 1$. To see this, note that Young's inequality implies

$$\|f \star K_1\|_{L^p} \le \|f\|_{L^1} \|K_1\|_{L^p}\,,$$

since $f \in L^1(\mathbb{R}^n)$ by assumption and $K_1(y) \le A/\|y\|^n$, for $\|y\| \ge 1$, is an L^p-function for $p > 1$. In summary, it then follows that

$$\|T_\varepsilon f - Tf\|_{L^p}^p = \int_{\mathbb{R}^n} \big|g_\varepsilon(x) - g(x)\big|^p\, dx \longrightarrow 0 \qquad (\varepsilon \longrightarrow 0)\,, \tag{1.45}$$

---

[3] Here we use also the fact that the constant C in (1.42) is independent of x, since f is compactly supported.

for all $f \in C_c^1(\mathbb{R}^n)$.

For general $f \in L^p(\mathbb{R}^n)$, we know that, for every $\delta > 0$, there exists by density $h \in C_c^1(\mathbb{R}^n)$ such that $\|f - h\|_{L^p} \leq \delta/3$. Moreover, due to (1.45), there exists $m_0 \in \mathbb{N}$ such that $\|T_{\varepsilon_m} h - T_{\varepsilon_n} h\|_{L^p} \leq \delta/3$, for every $m, n \geq m_0$. It follows that

$$\begin{aligned} \|T_{\varepsilon_m} f - T_{\varepsilon_n} f\|_{L^p} &\leq \|T_{\varepsilon_m} f - T_{\varepsilon_m} h\|_{L^p} + \|T_{\varepsilon_m} h - T_{\varepsilon_n} h\|_{L^p} + \|T_{\varepsilon_n} h - T_{\varepsilon_n} f\|_{L^p} \\ &\overset{(1.37)}{\leq} C\,\|f - h\|_{L^p} + \|T_{\varepsilon_m} h - T_{\varepsilon_n} h\|_{L^p} + C\,\|h - f\|_{L^p} \leq C\,\delta\,. \end{aligned}$$

Thus, the sequence $(T_\varepsilon f)_{\varepsilon > 0}$ is a Cauchy sequence and converges in L^p. Moreover, we denote the limit by $Tf \in L^p(\mathbb{R}^n)$ showing (1.38). – Note also that

$$\|Tf\|_{L^p} = \lim_{\varepsilon \to 0} \|T_\varepsilon f\|_{L^p} \overset{(1.37)}{\leq} C\,\|f\|_{L^p}\,.$$

□

Calderón-Zygmund Estimate for the Laplace Operator

From the previous theorem, we can now deduce the important so-called Calderón-Zygmund estimate for the Laplace operator. – Consider first the fundamental solution of the Laplace operator given by

$$\varGamma(x) = \frac{1}{n(2-n)\omega_n} \frac{1}{\|x\|^{n-2}}\,, \tag{1.46}$$

where ω_n is the volume of the n-dimensional unit ball and the dimension n is assumed to be larger or equal than two. By a straightforward computation, the first and second order partial derivatives reads as

$$\partial_i \varGamma(x) = \frac{1}{n\omega_n} \frac{x_i}{\|x\|^n}\,, \tag{1.47a}$$

$$\partial_j \partial_i \varGamma(x) = \frac{1}{n\omega_n} \frac{\big(\|x\|^2\,\delta_{ji} - n\,x_j x_i\big)}{\|x\|^{n-2}}\,, \tag{1.47b}$$

leading to the following estimates:

$$|\partial_i \varGamma(x)| \leq C\,\frac{1}{\|x\|^{n-1}}\,, \tag{1.48a}$$

$$|\partial_j \partial_i \varGamma(x)| \leq C\,\frac{1}{\|x\|^{n}}\,. \tag{1.48b}$$

Next, we define for $i, j = 1, \ldots, n$ the kernels

$$K_{ij}(x) = \partial_i \partial_j \varGamma(x)\,. \tag{1.49}$$

We claim that these kernels verify the hypothesis (1.35a)–(1.35c) of Theorem 1.11. – In order to see this, we first note that by a simple calculation

$$|\partial_l K_{ij}(x)| \leq C\,\frac{1}{\|x\|^{n+1}}\,.$$

Since the kernels K_{ij} are smooth on $\mathbb{R}^n\setminus\{0\}$, the mean value theorem implies that

$$\bigl|K_{ij}(x-y)-K_{ij}(x)\bigr| \leq \frac{C\,\|y\|}{\|x\|^{n+1}}\,.$$

Integration in polar coordinates then gives

$$\int_{2\|y\|\leq\|x\|}\bigl|K_{ij}(x-y)-K_{ij}(x)\bigr|\,dx \leq C\,\|y\|\int_{2\|y\|}^{\infty}\frac{1}{r^{n+1}}r^{n-1}dr \leq B\,,$$

showing that the Hörmander condition (1.35b) holds for K_{ij}. For the cancellation property (1.35c), assume first that $i\neq j$. Then, we have

$$\int_{R_1\leq\|x\|\leq R_2}K_{ij}(x)\,dx \overset{(1.47b)}{=} C\int_{R_1}^{R_2}\frac{1}{r^{n-2}}\left(\int_{S^{n-1}}x_ix_j\,d\sigma(x)\right)r^{n-1}\,dr\,.$$

This vanishes since the integral in parenthesis is zero. In the case of $i=j$, we observe that

$$\begin{aligned}\int_{R_1\leq\|x\|\leq R_2}K_{ii}(x)\,dx &\overset{(1.47b)}{=} \int_{R_1\leq\|x\|\leq R_2}\frac{\|x\|^2-n\,x_i^2}{\|x\|^{n-2}}\,dx\\ &= \int_{R_1\leq\|x\|\leq R_2}\frac{\|x\|^2-n\,x_l^2}{\|x\|^{n-2}}\,dx = \int_{R_1\leq\|x\|\leq R_2}K_{ll}(x)\,dx\,.\end{aligned}$$

Hence, we obtain that, for all $i=1,\ldots,n$,

$$n\int_{R_1\leq\|x\|\leq R_2}K_{ii}(x)\,dx = \int_{R_1\leq\|x\|\leq R_2}K_{11}+\ldots+K_{nn}\,dx\,.$$

The right-hand side being zero, we have thus shown that the cancellation property (1.35c) holds for the kernels K_{ij}. Because of (1.48b), the hypothesis (1.35a) also holds and the claim follows.

Take now $f\in C_c^1(\mathbb{R}^n)$ and define for $\varepsilon>0$

$$T_\varepsilon f(x) = \int_{\|y\|\geq\varepsilon}f(x-y)K_{ij}(y)\,dy\,. \tag{1.50}$$

From Theorem 1.11, we then deduce that

$$\|Tf\|_{L^p} \leq C\,\|f\|_{L^p}\,, \tag{1.51}$$

where $1<p<\infty$ and

$$Tf(x) = \lim_{\varepsilon\to0}T_\varepsilon f(x) = \int_{\mathbb{R}^n}f(x-y)K_{ij}(y)\,dy\,,$$

for all $x \in \mathbb{R}^n$ (see (1.44)).

In a next step, we consider $u \in C_c^3(\mathbb{R}^n)$ and the function f such that the Laplace equation

$$\Delta u = f \qquad \text{on } \mathbb{R}^n$$

holds. Obviously, we have that $f \in C_c^1(\mathbb{R}^n)$. The function u can be expressed with the help of the fundamental solution (1.46) as

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy + g(x)\,,$$

where g is an harmonic function on $\mathbb{R}^n$ which must tend to zero at infinity. By Liouville's theorem this implies that g must be identically zero. Moreover, we have that

$$\partial_i \partial_j u(x) = \int_{\mathbb{R}^n} \partial_i \partial_j \Gamma(x-y) f(y)\, dy\,, \tag{1.52}$$

for all $i, j = 1, \ldots, n$. For a proof of this we refer to []. Observing that the right-hand side of (1.52) is precisely the L^p-function Tf, we conclude that (1.51) translates to

$$\|\partial_i \partial_j u\|_{L^p} = \|Tf\|_{L^p} \leq C\, \|f\|_{L^p} = C\, \|\Delta u\|_{L^p}\,, \tag{1.53}$$

for $1 < p < \infty$. – For a general $u \in W^{2,p}(\mathbb{R}^n)$, there exists by density a sequence $(u_k)_{k\in\mathbb{N}}$ in $C_c^3(\mathbb{R}^n)$ such that $u_k \longrightarrow u$ in $W^{2,p}$, for $k \longrightarrow \infty$. From this, we deduce that

$$\|D^2 u\|_{L^p} = \lim_{k\to\infty} \|D^2 u_k\|_{L^p} \overset{(1.53)}{\leq} \lim_{k\to\infty} C\, \|\Delta u_k\|_{L^p} = C\, \|\Delta u\|_{L^p}\,.$$

In summary, we thus end up with the following Calderón-Zygmund estimate for the Laplace operator:

Theorem 1.12. *Let $u \in W^{2,p}(\mathbb{R}^n)$. Then, for $1 < p < \infty$, we have*

$$\|D^2 u\|_{L^p} \leq C \|\Delta u\|_{L^p}\,. \tag{1.54}$$

Example 1.13 (Counter-Example for L^1). On $\mathbb{R}^2$, we consider the function

$$f(x) = \frac{1}{\|x\|^2\, \log(1/\|x\|)^2}$$

being integrable over the disc $D_{1/2}$. We want to determine the function u which solves $\Delta u = f$ on $\mathbb{R}^2$. Since f is radial, we can assume the same for u implying that the Laplace equation reads in polar coordinates as

$$u''(r) + \frac{1}{r}\, u'(r) = \frac{1}{r^2\, \log(1/r)^2}\,.$$

Equivalently, we have

$$(r\,u')'(r) = \frac{1}{r^2\,\log(1/r)^2}\,.$$

Integration gives $u'(r) = 1/\bigl(r\,\log(1/r)\bigr)$ and a easy calculation

$$u''(r) = -\frac{1}{r^2\,\log(1/r)} + \frac{1}{r^2\,\log(1/r)^2}\,.$$

The first term, however, is *not* integrable over $D_{1/2}$.

Singular Integral Operators III

Next, we consider kernels K of the form

$$K(x) = \frac{\Omega(x)}{\|x\|^n}\,, \tag{1.55}$$

where Ω is an homogeneous function of degree 0, i.e., $\Omega(\delta x) = \Omega(x)$, for $\delta > 0$. In other words, the function Ω is radially constant and therefore completely determined by its values on the sphere S^{n-1}. Note also that K is homogeneous of degree $-n$, i.e., $K(\delta x) = \delta^{-n}K(x)$. – The following proposition shows how the conditions (1.35a)–(1.35c) on the kernel translate to kernels of the form (1.55).

Proposition 1.14. *Let $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function given by $K(x) = \Omega(x)/\|x\|^n$ with Ω an homogeneous function of degree 0 such that*

(i) The following cancellation property holds:

$$\int_{S^{n-1}} \Omega(x)\,d\sigma(x) = 0\,. \tag{1.56}$$

(ii) If we set

$$\omega(\delta) = \sup_{\substack{\|x-y\|\le\delta\\ x,y\in S^{n-1}}} \bigl|\Omega(x) - \Omega(y)\bigr|\,,$$

the following integral is finite:

$$\int_0^1 \frac{\omega(\delta)}{\delta}\,d\delta < \infty\,. \tag{1.57}$$

Then K satisfies the conditions (1.35a)–(1.35c).

Remark. a) Note that if Ω is Lipschitz on S^{n-1}, then $\omega(\delta) \le C\delta$ and the so-called Dini-type continuity condition (1.72) is full–filled. The same is true if Ω is assumed to be Hölder continuous with exponent γ on S^{n-1} since then $\omega(\delta) \le C\delta^\gamma$.

b) From the proposition, we conclude that Theorem 1.11 holds for kernels of the form (1.55) satisfying the two conditions (1.71) and (1.72).

Proof. The conditions (1.35a), respectively (1.35c), follow directly from (1.72), respectively (1.71) and integration in polar coordinates.

In order to establish (1.35b), we first observe that

$$\int_{2\|y\|\le\|x\|} \left|K(x-y)-K(x)\right| dx \le \int\limits_{2\|y\|\le\|x\|} \frac{\left|\Omega(x-y)-\Omega(x)\right|}{\|x-y\|^n}\, dx + \int\limits_{2\|y\|\le\|x\|} |\Omega(x)| \left|\frac{1}{\|x-y\|^n}-\frac{1}{\|x\|^n}\right| dx\,. \tag{1.58}$$

Since Ω is bounded due to (1.72) and as a consequence of the mean value theorem

$$\left|\frac{1}{\|x-y\|^n}-\frac{1}{\|x\|^n}\right| \le \frac{C\|y\|}{\|x\|^{n+1}}\,,$$

we conclude by integration in polar coordinates that the second integral on the right-hand side of (1.73) is finite. Note also that

$$\begin{aligned}\left|\Omega(x-y)-\Omega(x)\right| &= \left|\Omega\left(\frac{x-y}{\|x-y\|}\right)-\Omega\left(\frac{x}{\|x\|}\right)\right| \\ &\le \omega\left(\left\|\frac{x-y}{\|x-y\|}-\frac{x}{\|x\|}\right\|\right)\end{aligned}$$

by definition of the function ω. Moreover, if $2\|y\|\le\|x\|$, then $1/\|x-y\|^n \le C/\|x\|^n$ and also

$$\left\|\frac{x-y}{\|x-y\|}-\frac{x}{\|x\|}\right\| \le C\,\frac{\|y\|}{\|x\|}\,.$$

Inserting these estimates in the first integral on the right-hand side of (1.73), we obtain

$$\begin{aligned}\int\limits_{2\|y\|\le\|x\|} \frac{\left|\Omega(x-y)-\Omega(x)\right|}{\|x-y\|^n}\, dx &\le C \int\limits_{2\|y\|\le\|x\|} \frac{\omega\left(\frac{\|y\|}{\|x\|}\right)}{\|x\|^n}\, dx \\ &\le C\int_{2\|y\|}^{\infty} \frac{\omega\left(\frac{\|y\|}{r}\right)}{r^n}\, dr\,.\end{aligned}$$

Changing coordinates $\delta = C\|y\|/r$ and using (1.72), we deduce that the last integral is finite showing that (1.35b) holds. ☐

Example 1.15 (Riesz Transform). For $j=1,\dots,n$, we now consider the kernels $K_j(x)=\Omega_j(x)/\|x\|^n$ with

$$\Omega_j(x) = C_n\,\frac{x_j}{\|x\|}\,, \tag{1.59}$$

where $C_n =$. It is easy to check that Ω_j is Lipschitz on S^{n-1} and since Ω_j is an odd function the cancellation property

$$\int_{S^{n-1}} \Omega_j(x)\, d\sigma(x) = 0$$

also holds. For $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$, we then define the Riesz transform by

$$R_j f(x) = \lim_{\varepsilon \to 0} R_{j,\varepsilon} f(x)\,, \tag{1.60}$$

where

$$\begin{aligned} R_{j,\varepsilon} f(x) &= \int_{\varepsilon \le \|y\|} f(x-y) K_j(y)\, dy \\ &= C_n \int_{\varepsilon \le \|y\|} f(x-y) \frac{y_j}{\|y\|^{n+1}}\, dy\,. \end{aligned}$$

Note that the limit in (1.75) exists almost everywhere because of Theorem 1.17 below. Moreover, Theorem 1.11 implies that

$$\|R_j f\|_{L^p} \le C\, \|f\|_{L^p}\,, \tag{1.61}$$

for $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$. Computing the Fourier transform of $R_j f$, we obtain (see [])

$$\widehat{R_j f}(\xi) = \frac{i\,\xi_j}{\|\xi\|}\, \hat{f}(\xi)\,. \tag{1.62}$$

Now, we want to show that the Calderón-Zygmund estimate for the Laplace operator in Theorem 1.12 can also be established with the help of the Riesz transform. – For this purpose, let $f \in C^2_c(\mathbb{R}^n)$ and note that the Fourier transform of its second order partial derivatives are given by

$$\widehat{\partial_i \partial_j f}(\xi) = (i\,\xi_i)(i\,\xi_j)\, \hat{f}(\xi) = -\xi_i \xi_j\, \hat{f}(\xi)\,.$$

In particular, we have for the Fourier transform of the Laplace operator $\widehat{\Delta f}(\xi) = -\|\xi\|^2\, \hat{f}(\xi)$. This enables us to write the following:

$$\begin{aligned} \widehat{\partial_i \partial_j f}(\xi) &= -\xi_i \xi_j\, \hat{f}(\xi) = \frac{i\,\xi_i}{\|\xi\|} \frac{i\,\xi_j}{\|\xi\|}\, \widehat{\Delta f}(\xi) \\ &\overset{(1.76)}{=} \frac{i\,\xi_i}{\|\xi\|} \widehat{R_j(\Delta f)}(\xi) \overset{(1.76)}{=} \mathcal{F}\big(R_i(R_j(\Delta f))\big)(\xi)\,. \end{aligned}$$

Thus, we get

$$\partial_i \partial_j f = R_i\big(R_j(\Delta f)\big)\,. \tag{1.63}$$

From (1.61), it then follows that

$$\begin{aligned} \|\partial_i \partial_j f\|_{L^p} &= \big\|R_i\big(R_j(\Delta f)\big)\big\|_{L^p} \\ &\le C\, \|R_j(\Delta f)\|_{L^p} \le C\, \|\Delta f\|_{L^p}\,, \end{aligned}$$

for $1 < p < \infty$. Finally, by a density argument we recover (1.54).

The Critical Case $p = 1$

We want to emphasize that the singular integral convolution operator T is *not* bounded on $L^1(\mathbb{R}^n)$. This is confirmed by the following observation: If $0 \leq f \in L^1(\mathbb{R}^n)$ and $f \not\equiv 0$, then $Tf \notin L^1(\mathbb{R}^n)$. – To see this assume by contradiction that $Tf \in L^1(\mathbb{R}^n)$. Hence, its Fourier transform $\widehat{Tf}$ must be continuous. Since $0 \leq f \in L^1(\mathbb{R}^n)$ and $f \not\equiv 0$, note also that $\hat{f}(0) = \|f\|_{L^1} > 0$. On the other hand, we know that T can be realized by an homogeneous of degree 0 multiplier m, i.e., $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$. Consider, for example, the Riesz transform $R_j f$ with multiplier given by the right-hand side of (1.76). Since m is obviously not continuous at 0 and $\hat{f}(0) > 0$, we conclude that $\widehat{Tf}$ is also not continuous at 0 being a contradiction to the assumption $Tf \in L^1(\mathbb{R}^n)$.

However, as in the case of the Hardy-Littlewood maximal function, there is the following refinement:

Proposition 1.16. *Let B be a bounded subset of $\mathbb{R}^n$ and assume that*

$$\int_{\mathbb{R}^n} |f(x)|\big(1 + \log^+ |f(x)|\big)\, dx < \infty\,.$$

Then we have that $Tf \in L^1(B)$.

In order to prove this proposition, several estimates in the proof of Theorem 1.10 have to be formulated slightly differently. – Consider again $0 \leq |f| \in L^1(\mathbb{R}^n)$ and $\alpha > 0$ with the corresponding Calderón-Zygmund decomposition. Then, we introduce the positive function

$$\chi_f^\alpha(x) = \begin{cases} |f(x)| & \text{if } |f(x)| \leq \alpha \\ \alpha & \text{if } |f(x)| > \alpha\,. \end{cases} \tag{1.64}$$

This enables us to write, using the definition (1.22) for the function g,

$$\begin{aligned} \|g\|_{L^2}^2 &= \int_{\Omega^c} |g(x)|^2\, dx + \int_{\Omega} |g(x)|^2\, dx \\ &= \int_{\Omega^c} \big(\chi_f^\alpha(x)\big)^2\, dx + \int_{\Omega} |g(x)|^2\, dx \\ &\overset{(1.27)}{\leq} \int_{\mathbb{R}^n} \big(\chi_f^\alpha(x)\big)^2\, dx + C\,\alpha^2 \mu(\Omega)\,. \end{aligned} \tag{1.65}$$

Thus, it follows that (compare with (1.28))

$$\begin{aligned} \mu(\{x : |Tg(x)| > \alpha/2\}) &\leq \frac{C}{\alpha^2}\|g\|_{L^2}^2 \\ &\leq \frac{C}{\alpha^2} \int_{\mathbb{R}^n} \big(\chi_f^\alpha(x)\big)^2\, dx + C\mu(\Omega)\,. \end{aligned} \tag{1.66}$$

Moreover, we put (1.30) in the following form:

$$\begin{aligned}\int_{\tilde{\Omega}^c} |Tb(x)|\, dx &\leq B \sum_{k=1}^{\infty} \int_{C_k} |b_k(y)|\, dy \\ &\leq B \int_{\Omega} \big(|f(y)| + |g(y)|\big)\, dy \\ &\overset{(1.22)}{=} 2B \int_{\Omega} |f(y)|\, dy \overset{(1.10)}{\leq} 2B\, 2^n \alpha \mu(\Omega)\,.\end{aligned}$$

This then implies that (compare with (1.31))

$$\begin{aligned}\mu(\{x \in \mathbb{R}^n \,:\, |Tb(x)| > \alpha/2\}) &\leq \mu(\{x \in \tilde{\Omega}^c \,:\, |Tb(x)| > \alpha/2\}) + \mu(\tilde{\Omega}) \\ &\leq 4B\, 2^n \mu(\Omega) + (2\sqrt{n})^n\, \mu(\Omega) \\ &\leq C\, \mu(\Omega)\,. \end{aligned} \tag{1.67}$$

Combining (1.66) with (1.67), we end up with

$$\begin{aligned}\mu(\{x \,:\, |Tf(x)| > \alpha\}) &\leq \mu(\{x \,:\, |Tg(x)| > \alpha/2\}) + \mu(\{x \,:\, |Tb(x)| > \alpha/2\}) \\ &\leq \frac{C}{\alpha^2} \int_{\mathbb{R}^n} \big(\chi_f^{\alpha}(x)\big)^2\, dx + C\, \mu(\Omega) \end{aligned} \tag{1.68}$$

Now, we are ready to give a proof of Proposition 1.16.

Proof (of Proposition 1.16). The proof will be similar to the proof of Proposition 1.6. – We already know that

$$\|Tf\|_{L^1(B)} \leq \mu(B) + \int_1^{\infty} \lambda_{Tf}(\alpha)\, d\alpha\,.$$

Inserting (1.68) for $\lambda_{Tf}(\alpha)$, we deduce that

$$\begin{aligned}\|Tf\|_{L^1} \leq \mu(B) &+ \int_1^{\infty} \left(\frac{C}{\alpha^2} \int_{\mathbb{R}^n} \big(\chi_f^{\alpha}(x)\big)^2\, dx \right) d\alpha \\ &+ C \int_1^{\infty} \mu(\Omega_\alpha)\, d\alpha\,, \end{aligned} \tag{1.69}$$

where we changed slightly the notation for the cubes of the Calderón-Zygmund decomposition in order to emphasize that they depend on α.

Next, we compute the first integral on the right-hand side of the last equation

$$\begin{aligned}\int_1^{\infty} \left(\frac{C}{\alpha^2} \int_{\mathbb{R}^n} \big(\chi_f^{\alpha}(x)\big)^2\, dx \right) d\alpha &\overset{(1.64)}{=} C \int_{\mathbb{R}^n} \left(\int_0^{|f(x)|} d\alpha + \int_{|f(x)|}^{\infty} \frac{1}{\alpha^2} |f(x)|^2\, d\alpha \right) dx \\ &= C \int_{\mathbb{R}^n} \big(|f(x)| + |f(x)|\big)\, dx = 2C\, \|f\|_{L^1}\,.\end{aligned}$$

□

In Theorem 1.11, we have shown that the singular integral operation $Tf = \lim_{\varepsilon\to 0} T_\varepsilon f$ exists in the sense of L^p-convergence. The existence of this operation also in the sense of convergence almost everywhere is guaranteed by the next theorem.

Theorem 1.17. *Let $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function given by $K(x) = \Omega(x)/\|x\|^n$ with Ω an homogeneous function of degree 0 satisfying the hypothesis of Proposition 1.14. For $\varepsilon > 0$ and $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, we set*

$$T_\varepsilon f(x) = \int_{\|y\|\geq\varepsilon} f(x-y)\frac{\Omega(y)}{\|y\|^n}\,dy\,, \tag{1.70}$$

where the integral on the right-hand side is absolutely convergent for every x. Then, we have that $\lim_{\varepsilon\to 0} T_\varepsilon f(x)$ exists for a.e. $x \in \mathbb{R}^n$.

Remark. In the case of $f \in C_c^1(\mathbb{R}^n)$, the statement of the theorem was already an intermediate result in the proof of Theorem 1.11 (see (1.44)) and will be also needed to show the present general case.
Proof. □

Fractional Integral Operators

Recall that for $f \in C_c^1(\mathbb{R}^n)$ the function

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n} \Gamma(x-y)f(y)\,dy \\ &= \frac{1}{n(2-n)\omega_n}\int_{\mathbb{R}^n}\frac{1}{\|x-y\|^{n-2}}f(y)\,dy \end{aligned}$$

lies in $C^2(\mathbb{R}^n)$ and satisfies $\Delta u = f$. We also say that u is the Newtonian potential of f. Recall that its Fourier transform reads as

$$\hat{u}(\xi) = -\|\xi\|^{-2}\hat{f}(\xi)\,. \tag{1.71}$$

More generally, for $0 < \alpha < n$, we define the formal integral operators

$$I_\alpha f(x) = \frac{1}{\gamma(\alpha)}\int_{\mathbb{R}^n}\frac{1}{\|x-y\|^{n-\alpha}}f(y)\,dy\,, \tag{1.72}$$

where $\gamma(\alpha) =$. These will be called Riesz potentials of f or fractional integral operators. Note that in the case $\alpha = 2$, we recover the Newtonian potential in the sense that formally $\Delta I_2 f = f$ or equivalently $I_2 f = \Delta^{-1} f$. If f is now assumed to be a Schwartz function, then the following equality in the sense of distributions holds for the Fourier transform of the Riesz potentials:

$$\widehat{I_\alpha f}(\xi) = \|\xi\|^{-\alpha}\hat{f}(\xi)\,. \tag{1.73}$$

Comparing this with (1.71), we can roughly speaking say that the Riesz potential I_α defines negative fractional powers of the (negative) Laplace operator. We can write formally $I_\alpha f = (-\Delta)^{-\alpha/2} f$.

Next, we want to study the behavior of the Riesz potentials on L^p-spaces. – Assume that they are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, i.e., for $0 < \alpha < n$, we have

$$\|I_\alpha f\|_{L^q} \leq C \|f\|_{L^p} .$$

For such an estimate to be true, the exponent q cannot be arbitrary due to homogeneity considerations. More precisely, since $\big(I_\alpha f\big)_\delta(x) = \delta^\alpha I_\alpha f_\delta(x)$, where $f_\delta(x) = f(\delta x)$ denotes the function rescaled by δ, we get

$$\|I_\alpha f_\delta\|_{L^q} = \delta^{-\alpha} \|(I_\alpha f)_\delta\|_{L^q} = \delta^{-\alpha - \frac{n}{q}} \|I_\alpha f\|_{L^q} .$$

Applying (1.73) to the rescaled function f_δ, it follows that the exponent q must satisfy

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} . \tag{1.74}$$

Theorem 1.18 (Hardy-Littlewood-Sobolev Theorem for Fractional Integration). *Let $0 < \alpha < n$. Then, we have the following three statements:*

(i) For $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < n/\alpha$, the singular convolution integrals

$$\frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{1}{\|x-y\|^{n-\alpha}} f(y)\, dy$$

converge absolutely for almost every $x \in \mathbb{R}^n$.

(ii) Assuming that $1 < p < n/\alpha$, there exists a constant $C = C(n,p,q)$ such that

$$\|I_\alpha f\|_{L^q} \leq C \|f\|_{L^p} , \tag{1.75}$$

where the integrability exponent q is given by (1.74).

(iii) If $f \in L^1(\mathbb{R}^n)$, we have

$$\mu(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}) \leq \left(\frac{C \|f\|_{L^1}}{\lambda} \right)^{n/(n-\alpha)} , \tag{1.76}$$

for all $\lambda > 0$. In other words, the singular integral operators I_α are of weak type $(1,q)$ where $1/q = 1 - \alpha/n$.

Proof. First, we define $K(x) = 1/\|x\|^{n-\alpha}$ and hence

$$\int_{\mathbb{R}^n} \frac{1}{\|x-y\|^{n-\alpha}} f(y)\, dy = K \star f(x) .$$

We decompose the function K as a sum of an L^1-function K_1 and a bounded function K_∞ given by

$$K_1(x) = \begin{cases} K(x) & \text{if } \|x\| \le \varepsilon \\ 0 & \text{if } \|x\| > \varepsilon\,, \end{cases}$$

respectively, by

$$K_\infty(x) = \begin{cases} 0 & \text{if } \|x\| \le \varepsilon \\ K(x) & \text{if } \|x\| > \varepsilon\,, \end{cases}$$

In the decomposition $K = K_1 + K_\infty$, we have

$$K \star f(x) = K_1 \star f(x) + K_\infty \star f(x)\,.$$

Young's inequality then implies that

$$\|K_1 \star f\|_{L^1} \le \|K_1\|_{L^1} \|f\|_{L^p}\,,$$

for all $f \in L^p(\mathbb{R}^n)$. Denoting by p' the Hölder conjugate exponent to p, we observe that

$$\|K_\infty\|_{L^{p'}}^{p'} = \int_{\|x\| \ge \varepsilon} \left(\frac{1}{\|x\|^{n-\alpha}} \right)^{p'}$$

is finite, since from the assumption $p < n/\alpha$ it follows $n/(n-\alpha) < p'$. Using for the second convolution $K_\infty \star f$ again Young's inequality, we deduce

$$\|K_1 \star f\|_{L^\infty} \le \|K_1\|_{L^{p'}} \|f\|_{L^p}\,,$$

showing the first statement (i) of the theorem.

Let $\delta > 0$ and conclude from Hölder's inequality that

$$\begin{aligned} \int_{\|y-x\| \ge \delta} \frac{1}{\|x-y\|^{n-\alpha}} \,|f(y)|\, dy &\le C\, \|f\|_{L^p} \left(\int_\delta^\infty \left(\frac{1}{r^{n-\alpha}} \right)^{p'} r^{n-1}\, dr \right)^{1/p'} \\ &= C\, \|f\|_{L^p} \delta^{\alpha-(n/p)}\,. \end{aligned}$$

where $r = \|y - x\|$. Here we used that $p < n/\alpha$. Using (1.79a) in Lemma 1.19 below, we then obtain

$$|I_\alpha f(x)| \le C\Big(\delta^\alpha\, Mf(x) + \|f\|_{L^p} \delta^{\alpha-(n/p)} \Big)\,. \tag{1.77}$$

In order to minimize the right-hand side, we choose

$$\delta = \left(\frac{Mf(x)}{\|f\|_{L^p}} \right)^{-p/n}\,.$$

Inserting this in (1.77) gives the so-called Hedberg inequality

$$|I_\alpha f(x)| \le C\, Mf(x)^{1-(\alpha p/n)} \|f\|_{L^p}^{\alpha p/n}\,, \tag{1.78}$$

and also

$$|I_\alpha f(x)|^q \leq C\, Mf(x)^p \|f\|_{L^p}^{(\alpha p/n)q} .$$

Integrating over $\mathbb{R}^n$ and using the Hardy-Littlewood Maximal Theorem 1.4, we arrive at

$$\begin{aligned}\int_{\mathbb{R}^n} |I_\alpha f(x)|^q \, dx &\leq C \, \|f\|_{L^p}^{(\alpha p/n)q} \int_{\mathbb{R}^n} Mf(x)^p \, dx \\ &= C \, \|f\|_{L^p}^{(\alpha p/n)q} \|f\|_{L^p}^p .\end{aligned}$$

Thus, we end up with

$$\|I_\alpha f\|_{L^q} \leq C \, \|f\|_{L^p} ,$$

for $1 < p < n/\alpha$. □

Remark. Take a kernel $\bar{K} \in L^{n/(n-\alpha)}(\mathbb{R}^n)$. Then Young's inequality implies that

$$\|\bar{K} \star f\|_{L^{\bar{q}}} \leq \|\bar{K}\|_{L^{\frac{n}{n-\alpha}}} \|f\|_{L^p} ,$$

where

$$\frac{1}{\bar{q}} = \frac{n-\alpha}{n} + \frac{1}{p} - 1 = \frac{1}{p} - \frac{\alpha}{n} .$$

Note that $\bar{q}$ equals q defined in (1.74). – The singular kernels K defining the Riesz potentials, however, miss barely the regularity condition of being $L^{n/(n-\alpha)}$-functions and thus Young's inequality does not directly lead to the estimate (1.75).

Lemma 1.19. *Let $0 < \alpha < n$ and $\delta, \beta > 0$. Then, for all $x \in \mathbb{R}^n$, we have*

$$\int_{\|y-x\| \leq \delta} \frac{1}{\|x-y\|^{n-\alpha}} |f(y)| \, dy \leq C\delta^\alpha \, Mf(x) , \tag{1.79a}$$

$$\int_{\|y-x\| \geq \delta} \frac{1}{\|x-y\|^{n+\beta}} |f(y)| \, dy \leq \frac{C}{\delta^\beta} \, Mf(x) . \tag{1.79b}$$

Proof. We decompose the domain of integration in the following way:

$$\int_{\|y-x\| \leq \delta} \frac{1}{\|x-y\|^{n-\alpha}} |f(y)| \, dy = \sum_{k=0}^{\infty} \int_{\delta 2^{-(k+1)} \leq \|y-x\| \leq \delta 2^{-k}} \frac{1}{\|x-y\|^{n-\alpha}} |f(y)| \, dy .$$

Then, we compute

$$\begin{aligned}\int_{\|y-x\| \leq \delta} \frac{1}{\|x-y\|^{n-\alpha}} |f(y)| \, dy &\leq \sum_{k=0}^{\infty} \left(\frac{\delta}{2^{k+1}} \right)^{\alpha - n} \int_{\|y-x\| \leq \delta 2^{-k}} |f(y)| \, dy \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2} \right)^{\alpha-n} \left(\frac{\delta}{2^k} \right)^{\alpha} \left(\frac{\delta}{2^k} \right)^{-n} \int_{B_{\delta 2^{-k}}(x)} |f(y)| \, dy .\end{aligned}$$

The right-hand side can be written differently as

$$\omega_n \delta^\alpha \left(\frac{1}{2}\right)^{\alpha-n} \sum_{k=0}^{\infty} \left(\frac{1}{2^k}\right)^\alpha \frac{1}{\omega_n} \left(\frac{\delta}{2^k}\right)^{-n} \int_{B_{\delta 2^{-k}}(x)} |f(y)|\, dy\,.$$

Using the Definition 1.3 of the Hardy-Littlewood maximal function this is bounded by

$$\omega_n \delta^\alpha \left(\frac{1}{2}\right)^{\alpha-n} \sum_{k=0}^{\infty} \left(\frac{1}{2^k}\right)^\alpha Mf(x)\,,$$

showing (1.79a). □

The critical cases $p = 1$ and $p = n/\alpha$

In the case of $p = 1$, the exponent q equals $n/(n-\alpha)$ and we assume by contradiction that the following estimate holds:

$$\|I_\alpha f\|_{L^{\frac{n}{n-\alpha}}} \le C\, \|f\|_{L^1}\,. \tag{1.80}$$

Next, let $(\rho_k)_{k\in\mathbb{N}}$ be a mollifying sequence, i.e., the supports of the smooth functions ρ_k converge to the origin and $\|\rho_k\|_{L^1} = 1$, for all $k \in \mathbb{N}$. From (1.80), it then follows that

$$\|I_\alpha \rho_k\|_{L^{\frac{n}{n-\alpha}}} \le C\,,$$

for all $k \in \mathbb{N}$. Moreover, since $\rho_k \stackrel{k\to\infty}{\longrightarrow} \delta_0$ in $\mathcal{D}'$, we have that

$$I_\alpha \rho_k(x) \longrightarrow \frac{1}{\gamma(\alpha)} \frac{1}{\|x\|^{n-\alpha}} \qquad (k \longrightarrow \infty)\,,$$

for all $x \neq 0$. Applying dominated convergence, we conclude

$$\left\| \frac{1}{\gamma(\alpha)} \frac{1}{\|x\|^{n-\alpha}} \right\|_{L^{\frac{n}{n-\alpha}}} \le C\,.$$

This implies that the function $\|x\|^{-n}$ must be integrable over $\mathbb{R}^n$ which is obviously false. Thus the starting assumption (1.80) can not be true.

In the case of $p = n/\alpha$, we get $q = \infty$. For $\varepsilon > 0$, consider the function

$$f(x) = \frac{1}{\|x\|^\alpha \, \log(1/\|x\|)^{\frac{\alpha}{n}(1+\varepsilon)}}\, \chi_{B_{1/2}}\,.$$

It is not difficult to check that $f \in L^{n/\alpha}(\mathbb{R}^n)$. However, because

$$I_\alpha f(0) = \int_{B_{1/2}} \frac{1}{\|y\|^n \, \log(1/\|y\|)^{\frac{\alpha}{n}(1+\varepsilon)}}\, dy\,,$$

we deduce that the boundedness of $I_\alpha f$ fails near the origin if $\frac{\alpha}{n}(1+\varepsilon) \le 1$.

Theorem 1.20 (Sobolev Embedding Theorem). *Let* $1 \leq p < \infty$*,* k *a positive integer and*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}\,.$$

Then, we have the following three statements:

(i) If in addition $p < n/k$*, the embedding*

$$W^{k,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$$

is continuous. In the particular case of $k = 1$*, there exists a constant* $C = C(n,p)$ *such that*

$$\|f\|_{L^q} \leq C\,\|Df\|_{L^p}\,, \tag{1.81}$$

for all $f \in W^{1,p}(\mathbb{R}^n)$*.*

(ii)

(iii)

Remark. Theorem 1.18 can be interpreted as potential theoretic version of the Sobolev embedding theorem. Note that the latter is however also true for $p = 1$.

Proof. First, we consider the case $k = 1$. – Suppose that $f \in C_c^1(\mathbb{R}^n)$ and fix some $x \in \mathbb{R}^n$. Moreover, consider the curve $\gamma : [0,\infty] \longrightarrow \mathbb{R}^n$ given by $\gamma(r) = x + r\,\theta$ where $\theta \in \mathbb{R}^n$ is such that $\|\theta\| = 1$. Then the integral of the gradient ∇f of f over the curve γ equals

$$\int_0^\infty \nabla f\big(\gamma(r)\big)\cdot\gamma'(r)\,dr = \int_0^\infty \nabla f(x + r\,\theta)\cdot\theta\,dr = -f(x)\,, \tag{1.82}$$

since f has compact support by assumption. Integration over the unit sphere $S^{n-1}(x)$ centered at x leads to

$$f(x) = -\frac{1}{\omega_{n-1}}\int_{S^{n-1}(x)}\left(\int_0^\infty \nabla f(x + r\,\theta)\cdot\theta\,dr\right)d\sigma(\theta)\,.$$

This can also be written as

$$\begin{aligned} f(x) &= -\frac{1}{\omega_{n-1}}\sum_{i=1}^n\int_{S^{n-1}(x)}\left(\int_0^\infty \partial_i f(x + r\,\theta)\theta_i\,dr\right)d\sigma(\theta) \\ &= -\frac{1}{\omega_{n-1}}\sum_{i=1}^n\int_{S^{n-1}(x)}\left(\int_0^\infty \frac{\partial_i f(x + r\,\theta)\theta_i}{r^{n-1}}\,r^{n-1}\,dr\right)d\sigma(\theta) \end{aligned}$$

Next, we pass to rectangular coordinates $y = x + r\,\theta$ implying that

$$f(x) = \frac{1}{\omega_{n-1}}\sum_{i=1}^n\int_{\mathbb{R}^n}\frac{(x_i - y_i)}{\|x - y\|^n}\partial_i f(y)\,dy\,. \tag{1.83}$$

As direct consequence, we then have

$$\begin{aligned} |f(x)| &\leq C \sum_{i=1}^{n} \int_{\mathbb{R}^n} \frac{1}{\|x-y\|^{n-1}} |\partial_i f(y)|\, dy \\ &\leq C \sum_{i=1}^{n} I_1\big(|\partial_i f|\big)(x)\,. \end{aligned} \tag{1.84}$$

At this stage, we can apply Theorem 1.18 in the case $\alpha = 1$, in order to obtain

$$\|f\|_{L^q} \leq C \sum_{i=1}^{n} \|I_1\big(|\partial_i f|\big)\|_{L^q} \leq C \sum_{i=1}^{n} \|\partial_i f\|_{L^p}\,, \tag{1.85}$$

for $1/q = 1/p - 1/n$. The right-hand side being obviously bounded by the $W^{1,p}$-norm of f, the estimate (1.81) follows in the case of $f \in C_c^1(\mathbb{R}^n)$. $\square$

A

Before giving a detailed proof for the boundedness of the Fourier transform of the truncated kernels satisfying the three hypothesis of Theorem 1.11, we illustrate in the one-dimensional case how the boundedness fails if one of these hypothesis does not hold.

a) Consider the kernel $K(t) = 1/|t|$ and denote its truncation at $\varepsilon > 0$ by K_ε (see (A.6) below). It is not difficult to check that the Hörmander condition (1.35b) holds for K. The cancellation property (1.35c), however, is *not* satisfied.

As shown in (A.8) below, the Fourier transform of K_ε is given for a.e. $\xi \in \mathbb{R}$ by

$$\begin{aligned} \widehat{K_\varepsilon}(\xi) &= \lim_{k\to\infty} \int_{[-r_k, r_k]} e^{-i\,t\xi} K_\varepsilon(t)\, dt \\ &= \lim_{k\to\infty} \int_{\varepsilon\le|t|\le r_k} e^{-i\,t\xi} \frac{1}{|t|}\, dt\,. \end{aligned} \tag{A.1}$$

Next, we compute

$$\begin{aligned} \widehat{K_\varepsilon}(\xi) &= \lim_{k\to\infty} \int_{\varepsilon\le|t|\le r_k} \frac{\cos(\xi t) - i\,\sin(\xi t)}{|t|}\, dt \\ &= \lim_{k\to\infty} \int_{\varepsilon\le|t|\le r_k} \frac{\cos(\xi t)}{|t|}\, dt \\ &\overset{s=\xi t}{=} 2\,\mathrm{sgn}(\xi) \lim_{k\to\infty} \int_{\varepsilon|\xi|}^{r_k|\xi|} \frac{\cos(s)}{s}\, ds\,. \end{aligned} \tag{A.2}$$

Thus, we have to calculate an integral of the form $\int_\delta \infty \cos(s)/s\, ds$. For this purpose, the following decomposition is suitable:

$$\int_\delta^\infty \frac{\cos(s)}{s}\, ds = \int_\delta^{\pi/4} \frac{\cos(s)}{s}\, ds + \int_{\pi/4}^{\pi/2} \frac{\cos(s)}{s}\, ds + \sum_{k=0}^{\infty} \int_{k\pi+\pi/2}^{(k+1)\pi+\pi/2} \frac{\cos(s)}{s}\, ds\,. \tag{A.3}$$

For the first integral on the right-hand side, we have

$$\int_\delta^{\pi/4} \frac{\cos(s)}{s}\, ds \ge \frac{\sqrt{2}}{2} \int_\delta^{\pi/4} \frac{1}{s}\, ds\,,$$

which is logarithmic divergent in δ. The third integral on the right-hand side of (A.3), however, is convergent as monotone decreasing alternating sequence.

b) Now, we consider the kernel $K(t) = 1/t$. In opposite to the previous example a), the cancellation property (1.35c) is now also satisfied. Proceeding as before, we obtain (compare with(A.2))

$$\widehat{K_\varepsilon}(\xi) = -2i\,\mathrm{sgn}(\xi) \lim_{k\to\infty} \int_{\varepsilon|\xi|}^{r_k|\xi|} \frac{\sin(s)}{s}\, ds\,.$$

This integral, being of the form $\int_\delta^\infty \sin(s)/s\, ds$, can be splitted into

$$\int_\delta^\infty \frac{\sin(s)}{s}\, ds = \int_\delta^\pi \frac{\sin(s)}{s}\, ds + \sum_{k=1}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{\sin(s)}{s}\, ds\,. \tag{A.4}$$

Since the function $\sin(s)/s$ is continuous at zero, the first integral converges. The same holds for the second integral, by the same argument as before. – We have thus established that the Fourier transform of the kernel b) is uniformly bounded.

Remark. Roughly speaking, we can observe that the Hörmander condition full-filled by both kernels a) and b) ensures the convergence of the integral at infinity, whereas the cancellation property – only satisfied by the kernel b) – is responsible for the convergence at small distances.

Lemma A.1. *Let $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function such that*

$$|K(x)| \le \frac{A}{\|x\|^n}\ , \qquad \textit{for } \|x\| > 0\,. \tag{A.5a}$$

$$\int_{2\|y\|\le\|x\|} \big|K(x-y) - K(x)\big|\, dx \le B\ , \qquad \textit{for } \|y\| > 0\,. \tag{A.5b}$$

$$\int_{R_1<\|x\|<R_2} K(x)\, dx = 0\ , \qquad \textit{for } 0 < R_1 < R_2 < \infty\,. \tag{A.5c}$$

Moreover, for every $\varepsilon > 0$, we define

$$K_\varepsilon(x) = \begin{cases} K(x) & \textit{if } \|x\| \ge \varepsilon \\ 0 & \textit{if } \|x\| < \varepsilon\,. \end{cases} \tag{A.6}$$

Then, there exists a constant $C = C(n, A, B)$ such that

$$\|\hat{K}_\varepsilon\|_\infty \le C\,. \tag{A.7}$$

Proof. We first proof the lemma for the particular case $\varepsilon = 1$. – Note that because of (A.5a), the truncated kernel K_1 is an L^2-function. Defining $K_1^r(x) = K_1(x)\chi_{B_r}(x)$, dominated convergence implies that $K_1^r \overset{r\to\infty}{\longrightarrow} K_1$ in

L^2. By continuity of the Fourier transform, we also have that $\widehat{K_1^r} \stackrel{r\to\infty}{\longrightarrow} \widehat{K_1}$ in L^2. Hence, there exists a subsequence $(\widehat{K_1^{r_k}})_{k\in\mathbb{N}}$ such that $\widehat{K_1^{r_k}}(\xi) \stackrel{k\to\infty}{\longrightarrow} \widehat{K_1}(\xi)$, for a.e. $\xi \in \mathbb{R}^n$. More precisely, since K_1^r are L^1-functions having an integral representation for their Fourier transform, we have

$$\begin{aligned} \lim_{k\to\infty} \widehat{K_1^{r_k}}(\xi) &= \lim_{k\to\infty} \int_{\mathbb{R}^n} e^{-i\,x\cdot\xi} K_1(x)\chi_{B_r}(x)\,dx \\ &= \lim_{k\to\infty} \int_{B_{r_k}} e^{-i\,x\cdot\xi} K_1(x)\,dx = \widehat{K_1}(\xi)\,, \end{aligned} \tag{A.8}$$

for a.e. $\xi \in \mathbb{R}^n$.

In a next step, let $z \in \mathbb{R}^n$ and fix $\xi \in \mathbb{R}^n$. Consider then the function[1]

$$\begin{aligned} g(r) &= \left| \int_{B_r} e^{-i\,x\cdot\xi} K_1(x-z)\,dx - \int_{B_r(z)} e^{-i\,x\cdot\xi} K_1(x-z)\,dx \right| \\ &\leq \int_{B_r \Delta B_r(z)} \big|K_1(x-z)\big|\,dx\,. \end{aligned}$$

For $r \geq \|z\|$, it is not difficult to check that the symmetric difference $B_r \Delta B_r(z)$ is contained in an annulus with radii $r - \|z\|/2$ and $r + \|z\|/2$. Hence, there exists a constant C such that

$$\mu\big(B_r \Delta B_r(z)\big) \leq C\,\|z\| r^{n-1}\,. \tag{A.9}$$

This estimate becomes obvious for $r < \|z\|$. Moreover, in the case of $r \geq 2\|z\|$, we have

$$B_r \Delta B_r(z) \subset \mathbb{R}^n \setminus B_{r/2}(z)\,. \tag{A.10}$$

Thus, for every $x \in B_r \Delta B_r(z)$ with $r > 2\|z\|$, it follows

$$\big|K_1(x-z)\big| \stackrel{(A.5a)}{\leq} \frac{A}{\|x-z\|^n} \stackrel{(A.10)}{\leq} \frac{A}{(r/2)^n}\,.$$

From this it follows that

$$\begin{aligned} \lim_{r\to\infty} g(r) &\leq \lim_{r\to\infty} \int_{B_r \Delta B_r(z)} \big|K_1(x-z)\big|\,dx \\ &\leq \lim_{r\to\infty} \int_{B_r \Delta B_r(z)} \frac{A}{(r/2)^n}\,dx \stackrel{(A.9)}{\leq} \lim_{r\to\infty} 2^n AC\,\frac{\|z\| r^{n-1}}{r^n} = 0\,. \end{aligned}$$

More explicitly, the last result can be written as

$$\lim_{r\to\infty} \int_{B_r} e^{-i\,x\cdot\xi} K_1(x-z)\,dx = \lim_{r\to\infty} \int_{B_r(z)} e^{-i\,x\cdot\xi} K_1(x-z)\,dx\,. \tag{A.11}$$

[1] As usual, we denote the symmetric difference $E \setminus F \cup F \setminus E$ of the two sets E and F by $E \Delta F$.

Next, setting $w = x - z$ and using (A.8), we obtain that

$$\lim_{k\to\infty} \int_{B_{r_k}(z)} e^{-i\,x\cdot\xi} K_1(x-z)\,dx = \lim_{k\to\infty} e^{-i\,z\cdot\xi} \int_{B_{r_k}} e^{-i\,w\cdot\xi} K_1(w)\,dw$$
$$= e^{-i\,z\cdot\xi}\widehat{K_1}(\xi)\,,$$

for a.e. $\xi \in \mathbb{R}^n$. Inserting this in (A.11), we arrive at

$$\lim_{r\to\infty} \int_{B_r} e^{-i\,x\cdot\xi} K_1(x-z)\,dx = e^{-i\,z\cdot\xi}\widehat{K_1}(\xi)\,. \tag{A.12}$$

In particular, for every $\xi \in \mathbb{R}^n$, choosing

$$z = \pi\,\frac{\xi}{\|\xi\|^2} \tag{A.13}$$

such that $e^{-i\,z\cdot\xi} = -1$, it follows

$$\lim_{k\to\infty} \int_{B_{r_k}} e^{-i\,x\cdot\xi} K_1(x-\pi\,\xi/\|\xi\|^2)\,dx = -\widehat{K_1}(\xi)\,. \tag{A.14}$$

Combining (A.8) and (A.14), we end up with the following expression for the Fourier transform of K_1 at a.e. $\xi \in \mathbb{R}^n$:

$$\widehat{K_1}(\xi) = \lim_{k\to\infty} \frac{1}{2} \int_{B_{r_k}} e^{-i\,x\cdot\xi}\Big[K_1(x) - K_1(x-\pi\,\xi/\|\xi\|^2)\Big]\,dx\,. \tag{A.15}$$

With the help of the previous formula we will now show that $\widehat{K_1}$ is uniformly bounded on $\mathbb{R}^n$. – For this purpose, we first separate the integral on the right-hand side of (A.15) into two integrals I_1 and I_2 given by

$$I_1 = \int_{0\le\|x\|\le 2\pi/\|\xi\|} e^{-i\,x\cdot\xi}\Big[K_1(x) - K_1(x-\pi\,\xi/\|\xi\|^2)\Big]\,dx\,,$$

respectively, by

$$I_2 = \int_{2\pi/\|\xi\|\le\|x\|\le r_k} e^{-i\,x\cdot\xi}\Big[K_1(x) - K_1(x-\pi\,\xi/\|\xi\|^2)\Big]\,dx\,.$$

From the Hörmander condition (A.5b), we directly deduce that

$$I_2 \le \int_{2\pi/\|\xi\|\le\|x\|} \Big|K_1(x) - K_1(x-\pi\,\xi/\|\xi\|^2)\Big|\,dx \le B\,,$$

showing the boundedness of I_2. – It remains to give an upper bound for I_1 which is independent of ξ.

We write the integral I_1 as

$$
\begin{aligned}
I_1 &= I_3 - I_4 \\
&= \int_{0\le \|x\|\le 2\pi/\|\xi\|} e^{-i\,x\cdot\xi} K_1(x)\,dx - \int_{0\le \|x\|\le 2\pi/\|\xi\|} e^{-i\,x\cdot\xi} K_1(x-\pi\,\xi/\|\xi\|^2)\,dx\,.
\end{aligned}
$$

Obviously, by definition of K_1 in (A.6), we see that $I_3 = 0$ if $2\pi/\|\xi\| \le 1$. For the other case $2\pi/\|\xi\| > 1$, the cancellation property (A.5c) implies that

$$\int_{1\le \|x\|\le 2\pi/\|\xi\|} K_1(x)\,dx = 0\,.$$

This enables us to rewrite I_3 as

$$I_3 = \int_{1\le \|x\|\le 2\pi/\|\xi\|} \left[e^{-i\,x\cdot\xi} - 1\right] K_1(x)\,dx\,. \tag{A.16}$$

Since for the derivative of the smooth function $\phi(t) = e^{-i\,t}$, $t \in \mathbb{R}$, we have that $|\dot{\phi}(t)| = 1$, it follows that $|e^{-i\,t} - 1| \le |t|$. Inserting this into (A.16), we obtain

$$
\begin{aligned}
I_3 &\le \int_{1\le \|x\|\le 2\pi/\|\xi\|} \|\xi\|\,\|x\|\,|K(x)|\,dx \\
&\overset{(A.5a)}{\le} A\,\|\xi\| \int_{1\le \|x\|\le 2\pi/\|\xi\|} \frac{\|x\|}{\|x\|^n}\,dx\,.
\end{aligned}
$$

Using polar coordinates for the integration, the following estimate holds (recall that $2\pi/\|\xi\| > 1$):

$$I_3 \le CA\,\|\xi\| \int_1^{2\pi/\|\xi\|} d\rho = CA(2\pi - \|\xi\|) \le 4CA\,\pi\,. \tag{A.17}$$

Thus, in order to show the boundedness of I_1, it remains to bound the integral I_4.

Consider the integral

$$I_5 = \int_{\|x-z\|\le 2\pi/\|\xi\|} e^{-i\,x\cdot\xi} K_1(x-z)\,dx\,,$$

with z given by (A.13). By changing variables, we see that $I_5 = e^{-i\,z\cdot\xi} I_3$. Thus the estimate (A.17) implies

$$I_5 \le 4CA\,\pi\,. \tag{A.18}$$

On the other hand, since $2\|z\| = 2\pi/\|\xi\|$, it is easy to check that

$$|I_4 - I_5| \le \int_{B_{2\|z\|}\Delta B_{2\|z\|}(z)} \left|K_1(x-z)\right|dx\,.$$

Using the results (A.9) and (A.10) for the symmetric difference, we get that $\mu\big(B_{2\|z\|}\Delta B_{2\|z\|}(z)\big) \le C\|z\|^n$ and $B_{2\|z\|}\Delta B_{2\|z\|}(z) \subset \mathbb{R}^n \backslash B_{\|z\|}(z)$. From this, we conclude

$$\begin{aligned} |I_4 - I_5| &\le \int_{B_{2\|z\|}\Delta B_{2\|z\|}(z)} \big|K_1(x-z)\big|\,dx \\ &\overset{(A.5a)}{\le} \int_{B_{2\|z\|}\Delta B_{2\|z\|}(z)} \frac{A}{\|x-z\|^n}\,dx \\ &\le A\int_{B_{2\|z\|}\Delta B_{2\|z\|}(z)} \frac{A}{\|z\|^n}\,dx \le CA\,. \end{aligned} \tag{A.19}$$

Putting (A.18) and (A.19), we obtain that $I_4 \le |I_4 - I_5| + |I_5| \le CA + 4CA\,\pi$ and the boundedness of I_4 follows. – In summary, we have thus shown that

$$\|\widehat{K_1}\|_\infty \le C\,. \tag{A.20}$$

In a next step, we proof the lemma for general $\varepsilon > 0$[2]. – Let $\varepsilon > 0$ and define the new kernel $K'(x) = \varepsilon^n K(\varepsilon x)$. We claim that K' satisfy the conditions of the lemma. – First, we observe that

$$|K'(x)| = \varepsilon^n |K(\varepsilon x)| \overset{(A.5a)}{\le} \varepsilon^n \frac{A}{\|\varepsilon x\|^n} = \frac{A}{\|x\|^n}\,.$$

For the Hörmander condition, we compute

$$\begin{aligned} \int_{2\|y\|\le\|x\|} \big|K'(x-y) - K'(x)\big|\,dx &= \int_{2\|y\|\le\|x\|} \varepsilon^n \big|K(\varepsilon x - \varepsilon y) - K(\varepsilon x)\big|\,dx \\ &\overset{z=\varepsilon x}{=} \int_{2\varepsilon\|y\|\le\|z\|} \big|K(z-\varepsilon y) - K(z)\big|\,dz \overset{(A.5b)}{\le} B\,, \end{aligned}$$

and for the cancellation property

$$\begin{aligned} \int_{R_1<\|x\|<R_2} K'(x)\,dx &= \int_{R_1<\|x\|<R_2} \varepsilon^n K(\varepsilon x)\,dx \\ &\overset{z=\varepsilon x}{=} \int_{\varepsilon R_1<\|z\|<\varepsilon R_2} K(z)\,dz \overset{(A.5c)}{=} 0\,. \end{aligned}$$

This shows the claim.

Next, we define

$$K_1'(x) = \begin{cases} K'(x) & \text{if } \|x\| \ge 1 \\ 0 & \text{if } \|x\| < 1\,. \end{cases} \tag{A.21}$$

[2] The proof is straightforward for kernels of the particular form (1.55). In fact, since they are homogeneous of degree $-n$, we have that $\varepsilon^{-n}K_1(\varepsilon^{-1}x) = K_\varepsilon(x)$. Properties of the Fourier transform then directly imply that $\|\widehat{K_1}\|_\infty = \|\widehat{K_\varepsilon}\|_\infty$.

Comparing with (A.6), we directly get that $K_\varepsilon(x) = \varepsilon^{-n} K_1'(\varepsilon^{-1}x)$. From a well-known result for the Fourier transform, it then follows that

$$\widehat{K_\varepsilon}(\xi) = \widehat{K_1'}(\varepsilon\xi)\,.$$

As shown in the first part of the proof, the right-hand side is uniformly bounded (see (A.20)). This concludes the proof of the lemma. $\square$