Zbl 010.39103
Erdős, Paul
On primitive abundant numbers. (In English)
J. London Math. Soc. 10, 49-58 (1935).

The author has proved in a previous paper (see Zbl 010.10303) that the number $N(n)$ of primitive abundant numbers $\leq n$ is $O(n/\log^2 n)$. He proves in this paper the striking result that for large n,

$$ne^{-c_1,x} < N(n) < ne^{-c_2 x}$$

where $x = \sqrt{\log n \log\log n}$, and c_1, c_2 are absolute constants (say 8, $\frac{1}{2\,5}$
Define for any m

$$s_m = \prod_{\substack{p|m \\ p^2 \nmid m}} q_m = \frac{m}{s_m}.$$

The author proves that all but $O(ne^{-c_3 x})$ of the primitive abundant numbers $m \leq n$ satisfy: (1) $q_m < e^{\frac{1}{8}x}$, (2) the greates prime factor of m is $> e^x$. It is then shown that these primitive abundant numbers satisfy also (a) s_m has a divisor D_m between $\frac{1}{2}e^{\frac{1}{2}x}$ and $\frac{1}{2}e^{\frac{1}{8}x}$, (b) $2 \leq \sigma(m)/m < 2 + 2^{e^{-x}}$. [$\sigma(m) =$ sum of divisors of m]. Now, as in the previous paper, it follows that the numbers s_m/D_m are all different and $\leq 2ne^{-\frac{1}{8}x}$. This gives the upper bound for $N(n)$. To obtain the lower bound, the author considers numbers of the form $2^l p_1 ... p_k$, where $p_1, ... p_k$ are any k different primes between $(k-1)2^{l+1}$ and $k2^{l+1}$, and

$$e^{x-4} < 2^l < e^{x-3}, \qquad k = \left[\sqrt{\frac{\log n}{\log\log n}}\right] - 2.$$

These numbers are all primitive abundant, and an application of the prime-number theorem shews that there areat least ne^{-8x} of them.

Davenport (Cambridge)

Classification:
11A25 Arithmetic functions, etc.
11N25 Distribution of integers with specified multiplicative constraints