Zbl 103.27901
Erdős, Pál; Rado, R.
Intersection theorems for systems of sets (In English)
J. Lond. Math. Soc. 35, 85-90 (1960).

Let a and b be cardinals ≥ 1, let $\{X_\nu \mid \nu \in N\}$ be a family of sets of equal cardinality $|X_\nu| = c \leq b$, let $\Phi(a,b)$ be a cardinal, and consider the assertion (D): if $|N| > \Phi(a,b)$ then there exists a subset N' of N, $|N'| > a$, such that all $(X_\nu \cap X_\mu)$ are equal for $\nu \neq \mu, \nu, \mu \in N'$. Note that for finite $a, b = 1$, and $\Phi(a,b) = a^2$, this is a version of Dirichlet's box principle. It is extended here to theorem III: If a and b are finite then assertion (D) is valid for

$$\Phi(a,b) = b!a^{b+1}(1 - 1/2!a - \cdots - (b-1)/b!a^{b-1}).$$

The problem is left open whether or not the factor $b!$ may be replaced by d^b, for an appropriate constant d (the authors would have an application of this improved theorem III to number theory).
Theorem I(ii): If $a \geq 2$ and either a or b is infinite then assertion (D) is valid for $\Phi(a,b) = a^b$. Examples are produced to show that in I(ii) the estimating function a^b is the best possible, and in III the indicated Φ is best up to a factor of $b!$ (theorem II). I(ii) is a simple consequence of theorem I(i): Assertion (D) holds for $\Phi(a,b) = (b+1)b^b a^{b+1}$. However, the proof of I(i) uses the axiom of choice and a Ramification Lemma: Let α be an ordinal, $\{c_\gamma \mid \gamma < \alpha\}$ a family of cardinals. Let S be a set and for $\gamma < \alpha, \{s_\delta \mid \delta < \gamma\} \subseteq S$ let $M\{s_\delta\} \subseteq S$ and $|M\{S_\delta\}| \leq c_\gamma$. Let V be the set of all families $\{s_\delta \mid \delta < \alpha\}$ such that $s_\gamma \in M\{s_\delta \mid \delta < \gamma\}$ if $\gamma < \alpha$. Then $|V| \leq \prod_{\gamma<\alpha} c_\gamma$. Theorem III is also established by using I(i); of course the axiom of choice is not needed here, as I(i) and the Ramification Lemma enter in their finite version only. It should be noted that this is not the Ramification Lemma which is an extension of Königs's infinite lemma. It would be interesting to know what relation there is between these results and Ramsey's theorems.

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Classification:
05D05 Extremal set theory
04A99 Miscellaneous topics in set theory