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Erdős, Paul; Hajnal, András; Shelah, Saharon
On some general properties of chromatic numbers. (In English)
Topics in Topol., Colloqu. Keszthely 1972, Colloquia Math. Soc. Janos Bolyai 8, 243-255 (1974).

[For the entire collection see Zbl 278.00018.]

The starting point is the following Taylor problem [*W. Taylor*, Fundamenta Math. 71, 103-112 (1971; Zbl 238.02044); p. 106, problem 1.14]: What is the minimal cardinal number λ such that for every graph G with chromatic number $\chi G \geq \lambda$ and every cardinal $\sigma \geq \lambda$ there exists a graph G' such that $\chi(G') \geq \sigma$ and that G, G' have the same finite graphs? According to Taylor, $\lambda \geq \omega_1$, conjecturing $\lambda = \omega_1$. The authors formulate 7 other problems, prove 3 theorems and several lemmas. E.g., if $\chi(G) > \omega$, then for some $n < \omega$ the graph G contains odd circuits of length $2j+1$ for every $n < j < \omega$ (theorem 3). For any ordered set $(R, \prec)$ and any $i < \omega$ the authors define two sorts of graphs $G^0(R,i)$ for $i > 1$ and $G^1(R,i,t)$ for $i > 2$, $1 \leq t < i-1$ as $\prec$-increasing i-sequences of points of R such that

$$G^0(R,i) = \{(\varphi,\varphi') \mid \varphi(j+1) = \varphi'(j) \text{ for } j < i-1\}$$

for $i \geq 2$ and

$$G^1(R,i,t) = \{\varphi,\varphi') \mid \varphi(j+t) < \varphi'(t) < \varphi(j+t+1) < \varphi'(j+1) \text{ for } j < i-1-t\}$$

for $t \geq 3$; they put

$$S^0(i) = \psi(G^0(\omega,i),\omega) = \psi(G^0(R,i),\omega)$$

for $|R| \geq \omega$ and

$$S^1(i,t) = \psi(G^1(\omega,i,t),\omega) = \psi(G^1(R,i,t),\omega)$$

for $|R| \geq \omega$; the graphs S^0, S^1 are called "edge graphs" and Specker graphs respectively. Notations: For any cardinality $\tau \geq \omega$ let $B(\tau)$ be the system of all subgraphs of cardinality $< \tau$ of some complete graph with τ vertices; put $A(\tau) = P(B(\tau))$. If G is a given graph let

$$\psi(G,\tau) := \{G' \mid G' \in B(\tau), \quad G' \text{ being isomorphic to a subgraph of } G\}.$$

For $S \in A(\tau)$ let $G(S,\tau)$ be the class of graphs G satisfying $\psi(G,\tau) \subset S \in A(\tau)$; S is said τ-unbounded if for every cardinal λ there is some $G \in G(S,\tau)$ satisfying $\chi(G) > \lambda$. For a given operation F on cardinals satisfying $Fx \geq x^+$, the authors say that $S \in A(\omega)$ is ω- unbounded with the restriction F, if for every σ there is some $\lambda \geq \sigma$ and a graph G such that $\psi(G,\omega) \subset S$, $\chi(G) > \lambda$ and $|G| \leq F(\lambda)$. In particular, S is ω- unbounded with restriction ξ if S is so with the restriction F_ξ where

$$F_\xi(\lambda) = \kappa \Leftrightarrow \lambda = \omega_\alpha, \quad \kappa = \omega_{\alpha+1+\xi}.$$

Theorem 1: (α) $S^1(i,t)$ is ω-unbounded with restriction 0 for $2 \le i < \omega$; (β) $S^0(i)$ is ω-unbounded with restriction $\exp_{i-1}(\lambda)^+$ for $2 \le i < \omega$; (γ) $S^0(i)$ is not ω-unbounded with the restriction $\exp_{i-1}(\lambda)$ for $2 \le i < \omega$.

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Classification:
03E55 Large cardinals
03C68 Other classical first-order model theory
05C15 Chromatic theory of graphs and maps
05-02 Research monographs (combinatorics)